A brief introduction to the trace formula and its stabilization (1/2)

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Plan (1) Trace formula and its stab for G anisotropic red gp

(2) The same for G general.

(3) Application: Classical groups

G conn red $/\mathbb{Q}$, anisotropic

Assume (for simplicity)

- G_{der} derived subgp of G is simply conn
- G satisfies the Hasse principle.

O: doesn't satisfy; U: satisfies

§ Trace formula

interest

$$L^2([G]) = \widehat{\bigoplus_{\pi}} \pi^{\oplus \boxed{m(\pi)}}, \quad \text{finite } m(\pi)$$

$f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$, $\quad R(f) \curvearrowright L^2([G])$

action of right regular rep

$$\mathrm{tr}\, R(f) = \sum_{\pi} m(\pi) \cdot \mathrm{tr}(\pi(f)) \qquad \text{spectral info}$$

Write
$$[(Rf)\phi](x) = \int_{G(\mathbb{A})} f(g)\phi(gx)\,dg.$$
$$= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \underbrace{\sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma g)}_{K(x,g)\ \text{kernel operator}} \phi(g)\,dg$$

geometric info

$$\rightsquigarrow \quad \mathrm{tr}\, R(f) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} K(x,x)\,dx$$
$$= \sum_{\gamma \in [G(\mathbb{Q})]} \mathrm{vol}((G(\mathbb{Q})\backslash G(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x)\,dx.$$

$G_\gamma := \mathrm{Cent}_G(\gamma)$.

Thm (TF) $\quad \underbrace{\sum_{\pi} m(\pi)\, \mathrm{tr}\, \pi(f)}_{\text{spec side}} = \underbrace{\sum_{\gamma \in [G(\mathbb{Q})]} \tau(G_\gamma)\, O_\gamma(f)}_{\text{geom side}}$

invariant distributions

w/ Tamagawa measure & $\tau(G_\gamma)$: Tamagawa number.

§ Stable conjugacy

Def'n Let R be a $\mathbb{Q}$-alg ($R = \mathbb{Q}, \mathbb{Q}_v, \mathbb{A}$).

(1) Two elts $\gamma_1, \gamma_2 \in G(R)$ are called stably conjugate if $\gamma_1 \sim \gamma_2$ conj in $G(\bar{R})$, $\bar{R} = R \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$.

(2) The stable conj class of γ is $[\mathrm{Ad}(G(\bar{R}))\cdot\gamma] \cap G(R)$.

E.g. $G = SL_2 / \mathbb{R}$.

$$\gamma_1 = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$$

$\rightsquigarrow$ $\gamma_1 \sim \gamma_2$ in $SL_2(\mathbb{C})$ but not in $SL_2(\mathbb{R})$

$\Rightarrow$ $\gamma_1 \underset{\text{stab}}{\sim} \gamma_2$ stably conj.

Fact Given $\gamma_1, \gamma_2 \in G(R)$ stably conj. Let $g \in G(\bar{R})$, $g\gamma_1 g^{-1} = \gamma_2$. Then

(1) the class in $G(\bar{R})$ of

$\sigma \longmapsto g^{-1}\sigma g$ in $H^1(\Gamma, G_{\gamma_1}(\bar{R}))$ is indep of g,

$\left.\begin{matrix} \mathrm{Gal}_{\mathbb{Q}},\ R = \mathbb{Q}, \mathbb{A} \\ \mathrm{Gal}_{\mathbb{Q}_v},\ R = \mathbb{Q}_v \end{matrix}\right\} = \boxed{\Gamma} \ni \sigma$

$\uparrow$

$\bar{\mathbb{Q}}_v = \mathbb{Q}_v \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$

and will be denoted by $\mathrm{inv}(\gamma_1, \gamma_2)$.

(2) $\gamma_2 \longmapsto \mathrm{inv}(\gamma_1, \gamma_2)$ (fixing γ_1) is a bijection

$\{\text{stab conj class of } \gamma_1\} / G(R)$

$\longleftrightarrow \ker\big(H^1(\Gamma, G_{\gamma_1}(\bar{R})) \to H^1(\Gamma, G(\bar{R}))\big)$.

Def $R=\mathbb{Q}_v, \mathbb{A}$. $f\in C_c^\infty(G(\mathbb{A}))$, $\gamma\in G(R)$ semi-simple.

(1) $O_\gamma(f) := \int_{G_\gamma(R)\backslash G(R)} f(x^{-1}\gamma x)\,dx$

(2) When γ is regular,

$$SO_\gamma(f) := \int_{(G_\gamma\backslash G)(R)} f(x^{-1}\gamma x)\,dx = \sum_{\gamma'\underset{st}{\sim}\gamma} O_{\gamma'}(f).$$

(3) $d: C_c^\infty(G(R)) \longrightarrow \mathbb{C}$ is <u>stable</u>

if $d(f)=0$ whenever $SO_\gamma(f)=0$, $\forall\gamma$.

§ Pre-stabilization

$$V = V_{st}\oplus \mathcal{E},\quad \mathcal{E}=\bigoplus_H \mathcal{E}_H,\quad \text{with } \mathcal{E}_H \longrightarrow V_{st}^H.$$

Write $TF^G_{geom}(f) = \sum_{\gamma\in[G(\mathbb{Q})]} \tau(G_\gamma)\, O_\gamma(f)$

↑ geom side of TF

$$= \sum_{\gamma\in[\![G(\mathbb{Q})]\!]}\ \sum_{\substack{\gamma'\in[G(\mathbb{Q})]\\ \gamma'\underset{st}{\sim}\gamma}} \tau(G_{\gamma'})\,O_{\gamma'}(f)$$

$[\![\cdots]\!]$ stab classes

$[\cdots]$ conj classes

<u>Kottwitz</u> If γ,γ' are stab conj, then the Tanagawa numbers $\tau(G_\gamma)=\tau(G_{\gamma'})$.

$$\rightsquigarrow\ TF^G_{geom}(f) = \sum_{\gamma\in[\![G(\mathbb{Q})]\!]}\ \sum_{\gamma'\underset{st}{\sim}\gamma} \underbrace{O_{\gamma'}(f)}_{\text{adelic orb int.}}$$

By the fact before,

param the sum $\sum_{\gamma'\underset{st}{\sim}\gamma} O_{\gamma'}(f)$

$$\begin{tikzcd}
\text{not groups} \arrow[r] \arrow[dr, bend right] & \ker \arrow[r] \arrow[d, "\alpha"] & H^1(\Gamma, G_\gamma(\bar{\mathbb{Q}})) \arrow[r, "\beta_{\mathbb{Q}}"] \arrow[d, "\alpha"] & H^1(\Gamma, G(\bar{\mathbb{Q}})) \arrow[d] & \text{maps of sets} \\
 & \ker \arrow[r] & H^1(\Gamma, G_\gamma(\bar{\mathbb{A}})) \arrow[r, "\beta_{\mathbb{A}}"] \arrow[d] & H^1(\Gamma, G(\bar{\mathbb{A}})) \arrow[d] & \\
 & & \pi_1(G_\gamma)_{\Gamma,tor} & \pi_1(G)_{\Gamma,tor} &
\end{tikzcd}$$

by Kottwitz (later)

param what would like to have to make the orb int stable.

Assume γ regular ($\Rightarrow G_\gamma = T$ torus)

$$1 \to T(\bar{\mathbb{Q}}) \to T(\bar{\mathbb{A}}) \to T(\bar{\mathbb{A}})/T(\bar{\mathbb{Q}}) \to 1.$$

$$\rightsquigarrow \quad 1 \to Ш(T) \to H^1(\Gamma, T(\bar{\mathbb{Q}})) \to H^1(\Gamma, T(\bar{\mathbb{A}}))$$
$$\to H^1(\Gamma, T(\bar{\mathbb{A}})/T(\bar{\mathbb{Q}})) \underset{\substack{\uparrow \\ \text{Tate-Nakayama}}}{\cong} X_*(T)_{\Gamma,\mathrm{tor}}$$

<u>Borovoi</u> alg fundamental group $\pi_1(H)$
fin gen'd ab grp with Γ-action
for any conn red gp H over a field.
($T \subset H$ any max torus,
$\uparrow$ $\pi_1(H) := X_*(T)/X_*(T_{sc})$.)
Canonical, up to isoms.

<u>Kottwitz</u> $H/\mathbb{Q}$,
$$1 \to Ш(H) \to H^1(\Gamma, H(\bar{\mathbb{Q}})) \to H^1(\Gamma, H(\bar{\mathbb{A}})) \to \pi_1(H)_{\Gamma,\mathrm{tor}}$$ is exact

Def $\mathfrak{K}(\gamma) := \ker(\pi_1(G_\gamma)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}})^*$.

$$\rightsquigarrow \quad TF^G_{\mathrm{geom}}(f) = \sum_{\gamma \in [\![G(\mathbb{Q})]\!]} \tau(G_\gamma) \cdot \# Ш(G_\gamma)$$
$$\cdot |\mathfrak{K}(\gamma)|^{-1} \sum_{\kappa \in \mathfrak{K}(\gamma)} \sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \underset{st}{\sim} \gamma}} \kappa(\mathrm{inv}(\gamma,\gamma')) \cdot O_{\gamma'}(f)$$

$e(G_{\gamma'})$ Kottwitz's sign of centralizer

$$= \tau(G) \cdot \sum_{\gamma \in [\![G(\mathbb{Q})]\!]} \sum_{\kappa \in \mathfrak{K}(\gamma)} \underbrace{\sum_{\substack{\gamma' \in [G(\mathbb{A})] \\ \gamma' \underset{st}{\sim} \gamma}} \kappa(\mathrm{inv}(\gamma,\gamma')) \cdot {}^{\vee} O_{\gamma'}(f)}_{= O^\kappa_\gamma(f).}$$

Final result of pre-stabilization:

$$TF^G_{\mathrm{geom}}(f) = \tau(G) \cdot \sum_{\gamma \in [\![G(\mathbb{Q})]\!]} \underbrace{\sum_{\kappa \in \mathfrak{K}(\gamma)} O^\kappa_\gamma(f)}_{\text{"}SO_\gamma(f)\text{" when } \gamma = 1.}$$

Rmk Kottwitz proved that the global sign $e(G_\gamma) = 1$.

& when $\kappa = 1$.

$$\tau(G) \sum_{\gamma \in [G(\mathbb{Q})]} \underset{\overset{\parallel}{O^1_\gamma(f)}}{SO_\gamma(f)} = STF^G_{geom}(f).$$

§ Transfer

Bij: $(\gamma, \kappa) \longleftrightarrow (H, \kappa_H, \gamma_H)$

where (H, κ_H) = endoscopic elliptic grp data

$\gamma_H \in H(\mathbb{Q})$ (G,H)-reg elliptic.

H quasi-split $\mathbb{Q}$-grp

a very strict condition $\Big\{$ w/ fixed isom $T^G \xrightarrow{\sim} T^H$.

s.t. the Γ-str is twisted by Weyl grp.

κ clear of $\ker(\pi_1(H) \to \pi_1(G))_{\Gamma, tor}$.

Thm There exists $f \in C_c^\infty(H(\mathbb{A}))$ s.t.

$$O^\kappa_\gamma(f) = SO^H_{\gamma_H}(f^\kappa).$$

pf: Contains fund lemma

$\rightsquigarrow$ passing to Lie algs and use an observation of Waldspurger

$$\rightsquigarrow \quad TF^G_{geom}(f) = \tau(G) \cdot \sum_{\substack{(H,\kappa_H) \\ \overset{\cap}{E_{ell}(G)}}} |Out(H)|^{-1} \cdot \sum_{\substack{\gamma^H \in [H(\mathbb{Q})] \\ ell\ (G,H)\text{-reg}}} SO^H_{\gamma^H}(f^H).$$

ell endoscopic data

$$= \sum_{(H,\kappa_H) \in E_{ell}(G)} \iota(G,H) \cdot \underbrace{STF^H_{geom}(f^H)}_{\in V^H_{st}.}$$

§ Stabilization of the spectral side

$$TF^G_{spec}(f) = \sum_\pi m(\pi)\cdot tr(\pi(f)).$$

Remember G is still anisotropic.

Conj (Arthur) $L^2([G]) = \bigoplus_\psi \bigoplus_\pi \pi^{m(\psi,\pi)}$.

where $\psi : \mathcal{L}_{\mathbb{Q}} \times SL_2(\mathbb{C}) \longrightarrow {}^LG$

$\pi \in \boxed{\Pi_\psi} \leftarrow$ L-packet $\quad \prod_v \langle \pi_v, - \rangle$.

$$m(\psi,\pi) = mult(\mathcal{E}_\psi, \langle \pi, \overset{''}{-} \rangle)$$

$$= |S_\psi|^{-1} \sum_{x \in S_\psi} \mathcal{E}_\psi(x)\cdot \langle \pi, x \rangle^\vee.$$

When both S_ψ, S_{ψ_v} are abelian,

$$m(\psi,\pi) = \begin{cases} 0, & \langle \pi, - \rangle \neq \mathcal{E}_\psi, \\ 1, & \langle \pi, - \rangle = \mathcal{E}_\psi. \end{cases}$$

$$L^2([G]) = \bigoplus_\psi \bigoplus_{\pi : \langle \pi, - \rangle = \mathcal{E}_\psi} \pi .$$

$$\rightsquigarrow \; TF^G_{spec}(f) = \sum_\psi \sum_\pi |S_\psi|^{-1} \cdot \sum_{x \in S_\psi} e_\psi(x) \langle \pi, x \rangle tr\, \pi(f)$$

Take $S_\psi \ni s_\psi = \psi(1, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix})$
and consider the summand on RHS with $x = s_\psi$.

$$= \sum_\psi |S_\psi|^{-1} \sum_{x \in S_\psi} \mathcal{E}_\psi(x, s_\psi) \prod_v \underbrace{\sum_{\pi_v \in \Pi_{\psi_v}} \langle \pi_v, x \cdot s_\psi \rangle tr\, \pi_v(f_v)}.$$

Using Bij: $(\psi, s) \longleftrightarrow (H, \mathcal{K}, \psi^H)$,

$$\rightsquigarrow \; \sum_{\pi_v \in \Pi_{\psi_v}} \langle \pi_v, x \cdot s_\psi \rangle tr\, \pi_v(f_v) = \sum_{\pi^H_v \in \Pi_{\psi^H_v}} \langle \pi^H_v, s_{\psi^H} \rangle \cdot tr\, \pi^H_v(f^H_v).$$

$$\overset{\text{def'n}}{=:} S\Theta^H_{\psi^H_v}(f^H_v)$$

Stable character of the parameter ψ^H_v.

$$\rightsquigarrow \; TF^G_{spec}(f) = \sum_{(H,\mathcal{K}) \in \mathcal{E}_{ell}(G)} \iota(G,H) \cdot \sum_{\psi^H} \mathcal{E}_{\psi^H}(s_{\psi^H})\, S\Theta_{\psi^H}(f^H).$$

$\Rightarrow$ get the stabilization of spectral side.

<u>Conj</u> (Stable multiplicity formula, Arthur).

$$STF^G_{geom}(f) = STF^G_{spec}(f) := \sum_{\psi} \varepsilon_\psi(s_\psi)\, S\Theta_\psi(f).$$