A brief introduction to the trace formula and its stabilization (2/2)

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G general red grp / $\mathbb{Q}$.

(1) Non-invariant trace formula

(2) Invariant trace formula

(3) Stable trace formula

(4) Application: classical groups.

Issues when G is isotropic

(1) $L^2([G])$ no longer decomposes discretely

$$L^2([G]) = \underbrace{L^2_{disc}([G])}_{\text{have trace}} \oplus \underbrace{L^2_{cont}([G])}_{\text{no trace}} \circlearrowleft R(f)$$

In general, $L^2_{cont}([G]) = \bigoplus\limits_M \mathrm{Ind}_P^G\big(\overbrace{L^2_{disc}([M])}^{= \int^{\oplus}_{i\mathfrak{a}^*_M} L^2_{disc}([M]^1)}\big)^{\omega_G(M)}$

realized by theory of Eisenstein series.

(2) On geom side, $G(\mathbb{Q})$ will include many more elts

(G anisotropic: ss + ell only.)

Have: ss / non-ell elts: $\mathrm{vol}([G_\gamma]) = \infty$,

unipotent elts: $O_\gamma(f) = \infty$.

§ Truncation and non-invariant trace formula

Define $\int \underbrace{k^T(x,x)}_{\text{truncated kernel}} dx =: J^T(f)$ distribution.

with $\sum\limits_{o \in \mathcal{O}} J^T_o(f) = J^T(f)$, $\mathcal{O}$ = equiv class of $\gamma \in G(\mathbb{Q})$.

$$\sum_{\chi \in \mathfrak{X}} J_\chi^T(f) = J^T(f), \quad \mathfrak{X} = \text{cuspidal data}$$

$$= \left\{ (P, \sigma) \;\middle|\; \begin{array}{l} P \subset G \text{ parabolic}, \\ \sigma \text{ cusp aut rep of } M_P(\mathbb{A}) \end{array} \right\}$$

Called "coarse expansion" of $J^T(f)$.

where $T := T_0$ magic value

(which is often zero, e.g. when $G = GL_n, SL_n$, etc.)

- If $\mathfrak{o}$ consists of reg ss ell elts

  then $\quad J_{\mathfrak{o}}(f) = \mathrm{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})^1) \cdot \int_{G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})} f(x^{-1}\gamma x)\, dx$

- If $\chi = (G, \sigma) \in \mathfrak{X}$,

  then $\quad J_\chi(f) = \mathrm{multi}(G) \cdot \mathrm{tr}\, \sigma(f)$.

E.g. $G = SL_2$, $\mathfrak{o}$ = set of all unipotent elts

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \cdot f(1)$$

const term in Laurent exp at $s=0$

$$+ (\text{fin part at } s=0 \text{ of } Z(s)).$$

with $\quad Z(s) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x) \cdot e^{-s \cdot H_B(x)} \cdot dx \;, \quad \gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$

where $\quad H_B : G(\mathbb{A}) \longrightarrow \mathbb{R}$,

$$G(\mathbb{A}) = N(\mathbb{A}) \cdot \underbrace{\boxed{T(\mathbb{A})}}_{=\mathbb{A}^\times} \cdot \underbrace{K}_{=SO(2,\mathbb{R}) \cdot SL_2(\hat{\mathbb{Z}})}$$

$SO(2,\mathbb{R}) \cdot SL_2(\hat{\mathbb{Z}})$ max cpt subgrp

s.t. $\quad H_B(n \cdot t \cdot k) = \log |t|$.

Observe $\quad Z(0) = \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma x)\, dx = \prod_v \underbrace{O_\gamma(f_v)}_{=\zeta_v(1)} \sim \zeta(1) = \infty$.

<u>Fine expression</u>

$$\lim_{S} \sum_M |W_0^M| \cdot |W_0^G|^{-1} \sum_{\gamma \in G(\mathbb{Q})/G(\mathbb{A}_S)} \underbrace{a^M(\gamma)}_{\text{global coeff}} \cdot \underbrace{J_M(\gamma, f)}_{\text{local dist.}}$$

$$= \sum_M |W_0^M| \cdot |W_0^G|^{-1} \cdot \int_\sigma a^M(\sigma) \cdot J_M(\sigma, f) \quad \text{fine exp.}$$

Take $\gamma \in M(\mathbb{Q})$ ss elt, $M_\gamma = G_\gamma$, (G,M)-regular
s.t. M_γ not contained in a smaller Levi subgrp.

$\rightsquigarrow$ $a^M(\gamma) = \mathrm{vol}([M_\gamma]')$,

$$J_M(\gamma, f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(x^{-1}\gamma x) \underbrace{v_M(x)}_{\text{weight factor.}} dx$$

weighted orbital integral.

Ex. $\chi = (M, \sigma)$, σ cusp aut rep of M.

$\mathrm{Stab}(W_G(M), \sigma) = M$.

$$\Rightarrow \quad a^M(\gamma) = \mathrm{mult}(\sigma), \quad J_M(\sigma, f) = \int_{i\mathfrak{a}_M^*} \mathrm{tr}(\underbrace{\mathcal{M}_P(\sigma_\lambda)}_{\text{an operator-valued weight factor}} \cdot I_P^G(\sigma_\lambda, f))\, d\lambda$$

Ex $G = SL_2$, $\left.\frac{d}{ds}\right|_{s=0} s \cdot Z(s) = ?$

Fix S large finite set of places

$\rightsquigarrow$ $Z(s) = Z_S(s) \cdot Z^S(s)$.

$$\& \quad \underbrace{\boxed{Z_S(0)}}_{J_G(u,f)} \cdot \underbrace{\boxed{\left.\frac{d}{ds}\right|_{s=0} (s \cdot Z^S(s))}}_{a^G(u)} + \underbrace{\boxed{\left.\frac{d}{ds}\right|_{s=0} Z_S(s)}}_{J_T(1,f)} \cdot \underbrace{\boxed{(s \cdot Z^S(s))|_{s=0}}}_{a^T(1).}$$

Ex (on spectral side)

SL_2, (T, μ), (G,M)-reg $\Leftrightarrow$ $\mu^2 \neq 1$.

In (G,M)-reg case: $\int_{-\infty}^{+\infty} \mathrm{tr}(M_P(it)^{-1} \cdot M_P'(it) \cdot I_P^G(\mu_{it}, f))\, dt$

In non-(G,M)-reg case: $\int_{-\infty}^{+\infty} \mathrm{tr}(M_P(it)^{-1} \cdot M_P'(it) \cdot I_P^G(\mu_{it}, f))\, dt$

$+ \mathrm{tr}(M_P \circ I_P^G(\mu, f))$.

§ The discrete part of the spectral side

$$I^G_{disc}(f) = \sum_M |W_0^M| \cdot |W_0^G|^{-1} \cdot \sum_{w \in \boxed{(W_M^G)_{reg}}} |\det(w-1|\mathfrak{a}_M^G)|^{-1} \cdot \mathrm{tr}(M_P(w) \cdot \mathcal{I}_P^G(f))$$

$\hookrightarrow \mathfrak{a}_M^G = X_*(A_M/A_G) \otimes_{\mathbb{Z}} \mathbb{R}$

<u>note</u> $M = G$: $\mathrm{tr}(R(f) \mid L^2_{disc}([G]))$.

§ The invariant trace formula

<u>Thm</u> The map $\phi_M^G : f \longmapsto I_M^G(\cdot, f)$

is well-def'd cont $\mathcal{H}(G(\mathbb{A}_S)) \to \mathcal{I}(M(\mathbb{A}_S))$.

There exist unique invariant distributions

$$I_M^G(\gamma, f), \quad I_G^M(\pi, f)$$

& $$J_M^G(\gamma, f) = \sum_{\substack{M \subset L \subset G \\ \text{Levi}}} I_M^L(\gamma, \phi_L^G(f)),$$

$$J_M^G(\pi, f) = \sum_{\substack{M \subset L \subset G \\ \text{Levi}}} I_M^L(\pi, \phi_L^G(f)).$$

<u>Thm</u> We have the decomp

$$J^G(f) = \sum_{L \subset G} |W_0^L| \cdot |W_0^G|^{-1} \cdot I^L(\phi_L^G(f)).$$

Have $\sum_M |W_0^M| \cdot |W_0^G|^{-1} \sum_\gamma a^M(\gamma) \cdot I_M(\gamma, f) = I(f)$

$= \sum_M |W_0^M| \cdot |W_0^G|^{-1} \int_\pi a^M(\pi) I_M(\pi, f)$.

§ The stable trace formula

<u>Thm</u> We have a decomp

autom transfer of f.

$$I^G(f) = \sum_{G'} \iota(G, G') \cdot S^{G'}(f')$$

where S^G is a stable dist s.t.

$$\sum_M |W_0^M|\cdot|W_0^G|^{-1}\sum_\delta b^M(\delta)\cdot S_M^G(\delta,f)$$
$$=\sum_M |W_0^M|\cdot|W_0^G|^{-1}\int_\phi b^M(\phi)\cdot S_M^G(\phi,f)\,d\phi.$$

$\underline{\text{Ex}}$ $G=SL_2$, $I_{orb}^G(f)=\sum_\gamma a^G(\gamma)\cdot I_G(\gamma,f)$.

reg ss ell elt.

unipotent elts

Last time: $I_{ell}^G(f)=\sum_{G'} \iota(G,G')\cdot S_{ell,\underbrace{(G,G')-reg}}^{G'}(f')$.

resp: + unip   anisotropic tori   unconditional if $G=G'$.

$S^G(f)+\sum_{G\neq G'}\iota(G,G')\,S_{(G,G')-reg}^{G'}(f')$.

resp. + sing.

§ Application to classical groups

- TF + stab for G classical.
- TF + stab for twisted GL_N.

} endoscopic grps are products of classical grps

$\underline{\text{Formal parameters}}$ irreps of $\mathcal{L}_\mathbb{Q}$ $\longleftrightarrow$ cusp autom reps of GL_N.

$\Psi(N)$ = formal parameters for GL_N
$\cup$
$\Psi(G)$ = formal parameters for G

For $\psi\in\Psi(N)$ $\rightsquigarrow$ infinitesimal char ($v|\infty$)
Satake parameters ($v|p$).

$$L^2_{disc}([G])=\bigoplus_{\psi\in\Psi(N)} L^2_{disc,\psi}([G]).$$

$$I^G_{disc}(f)=\sum_\psi I^G_{disc,\psi}(f)$$

$$I^G_{disc,\psi}(f)=\sum_{G'}\iota(G,G')\cdot S^{G'}_{disc,\psi}(f').$$

§ Extracting the payload from $I^G_{\mathrm{disc},\psi}$.

Thm Let $\psi \in \Psi(N)$, G either classical or twisted GL_N.

$$I^G_{\mathrm{disc},\psi}(f) - \mathrm{tr}\, R_{\mathrm{disc},\psi}(f)$$
$$= |S_\psi|^{-1} \sum_{x \in S_\psi} \varepsilon^G_\psi(x)\, |W^0_\psi|^{-1} \sum_{w \in (W^M_G)_{\mathrm{reg}}} S^0_\psi(w) \cdot |\det(w-1)|^{-1} \cdot f_G(\psi, u).$$

Key object: $f_G(\psi, u)$

$$f_G(\psi, u) = \sum_{\pi \in \Pi_\psi} \mathrm{tr}(R_P(w, \pi, \psi) \cdot I^G_{P,\psi}(f))$$

RHS of Arthur's Conj

Define $^\circ r_{\mathrm{disc},\psi} = \mathrm{tr}\, R_{\mathrm{disc},\psi} - \text{expectation}$

Cor $I^G_{\mathrm{disc},\psi}(f) - {}^\circ r_{\mathrm{disc},\psi}$

$$= |S_\psi|^{-1} \sum_{x \in S_\psi} i_\psi(x)\, \varepsilon^G_\psi(x) \cdot f_G(\psi, u).$$

where

$$i_\psi(x) = |W_0|^{-1} \sum_{w \in W_{\mathrm{reg}}} S^0_\psi(w) \cdot |\det(w-1)|^{-1}.$$

Thm Let $\psi \in \Psi(N)$, G classical or twisted GL_N.

$$I^G_{\mathrm{disc},\psi}(f) - \sum_{G' \text{ simple}} S^{G'}(f')$$
$$= |S_\psi|^{-1} \sum_{x \in S_\psi} \varepsilon^G_\psi(x \cdot s_\psi) \sum_{s \in \Sigma_{\mathrm{ell}}(x)} |\pi_0(\bar{S}_\psi, s)|^{-1} \cdot \sigma(\bar{S}^0_\psi)\, f'_G(\psi, s).$$

Rmk Recall $\mathrm{Bij}: (\psi, s) \longleftrightarrow (H, \kappa_H, \psi^H)$

$$\rightsquigarrow f'_G(\psi, s) = S\Theta_{\psi'}(f').$$

$$^\circ S_{\mathrm{disc},\psi} := S^{G'}(f) - \text{expectation}$$

Cor $I^G_{\mathrm{disc}}(f) - {}^\circ S_{\mathrm{disc},\psi}(f) = |S_\psi|^{-1} \sum_{x \in S_\psi} e_\psi(x) \cdot \varepsilon^G_\psi(x) \cdot f'_G(\psi, x \cdot s_\psi).$

Known Combinatorially, $\epsilon_\psi(x) = e_\psi(x)$.

Conj (Global intertwining relation)

$$f_G(u, \psi) = f'_G(x \cdot s_\psi, \psi).$$

Conditional on:

(1) (Twisted) weighted fundamental lemma

(2) Results for "simple" non-generic local parameters [A27]

(3) [A26]: Property of normalized intertwining operators for twisted GL_N.

(4) Compatibility of twisted transfer with Aubert involution.