Geometric and arithmetic theta correspondences

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Overview

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For dual pairs $(G, H) = (\mathrm{Sp}(W), \mathrm{O}(V))$, $(\mathrm{U}(W), \mathrm{U}(V))$...

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- F/F_0: quadratic extension of number fields.
- W: the standard split skew-hermitian space of dimension $2n$ over F, i.e., $W = F^{2n}$ with skew-hermitian matrix $w_n = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$.
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- $H = \mathrm{U}(V)$: a unitary group over F_0 in m-variables ("$= \mathrm{U}(m)$").
- $P = MN \subseteq G$: the standard Siegel parabolic subgroup stabilizing the maximal isotropic subspace $F^n \oplus 0^n \subseteq W = F^{2n}$.

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \right\},$$

$$N = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} : b \in \mathrm{Herm}_n \right\}.$$

Weil representation

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- $\mathbb{A} = \mathbb{A}_{F_0}$: the ring of adeles of F_0.
- $\eta : \mathbb{A}^\times / F_0^\times \to \mathbb{C}^\times$: the quadratic character associated to F/F_0.
- Fix $\chi : \mathbb{A}_F^\times \to \mathbb{C}^\times$ a character such that $\chi|_{\mathbb{A}^\times} = \eta^m$.
- $\mathbf{W} := \mathrm{Res}_{F/F_0}(V \otimes_F W)$.
- χ determines a splitting homomorphism

$$G(\mathbb{A}) \times H(\mathbb{A}) \to \mathrm{Mp}(\mathbf{W}_{\mathbb{A}})$$

  lifting the natural homomorphism $G(\mathbb{A}) \times H(\mathbb{A}) \to \mathrm{Sp}(\mathbf{W}_{\mathbb{A}})$.
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- (χ, ψ) gives rise to a Weil representation $\omega = \omega_{\chi,\psi} = \otimes_v \omega_{\chi_v,\psi_v}$ of $G(\mathbb{A}) \times H(\mathbb{A})$.
- ω has an explicit realization on $\mathscr{S}(V(\mathbb{A})^n)$, the space of Schwartz functions, known as the Schrödinger model: for $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$ and $\mathbf{x} \in V(\mathbb{A})^n$,

$$\begin{aligned}
\omega(m(a))\varphi(\mathbf{x}) &= \chi(\det a)|\det a|_F^{m/2}\varphi(\mathbf{x}\cdot a), && m(a) \in M(\mathbb{A}),\\
\omega(n(b))\varphi(\mathbf{x}) &= \psi(\mathrm{tr}\, b\,(\mathbf{x},\mathbf{x}))\varphi(\mathbf{x}), && n(b) \in N(\mathbb{A}),\\
\omega(w_n)\varphi(\mathbf{x}) &= \gamma_{\mathbb{V}}^n \cdot \widehat{\varphi}(\mathbf{x}), && w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},\\
\omega(h)\varphi(\mathbf{x}) &= \varphi(h^{-1}\cdot \mathbf{x}), && h \in H(\mathbb{A}).
\end{aligned}$$

  - $(\mathbf{x},\mathbf{x}) = ((x_i, x_j))_{1\le i,j\le n} \in \mathrm{Herm}_n(\mathbb{A})$ is the moment matrix.
  - $\widehat{\varphi}$ is the Fourier transform of φ.

Classical theta lifting

Classical theta lifting

- Associated to $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$, define the (two-variable) theta function

$$\theta(g, h, \varphi) := \sum_{\mathbf{x} \in V^n} \omega(g, h)\varphi(\mathbf{x}) = \sum_{\mathbf{x} \in V^n} \omega(g)\varphi(h^{-1}\mathbf{x}), \quad g \in G(\mathbb{A}), h \in H(\mathbb{A}).$$

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- Using $\theta(g,h,\varphi)$ as an integral kernel allows one to lift automorphic forms on G to automorphic forms on H (and vice versa).
- For a cuspidal automorphic form $\phi \in \mathscr{A}(G(\mathbb{A}))$, define the theta lift $\theta_\varphi(\phi)$ of ϕ:

$$\theta_\varphi(\phi)(h) := \langle \theta(-,h,\varphi), \phi\rangle_G = \int_{[G]} \theta(g,h,\varphi)\overline{\phi(g)}\mathrm{d}g.$$

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- π: a cuspidal automorphic representation of $G(\mathbb{A})$.
- Get an $G(\mathbb{A})\times H(\mathbb{A})$-equivariant linear map
$$\theta : \mathscr{S}(V(\mathbb{A})^n)\otimes \pi^\vee \to \mathscr{A}(H(\mathbb{A})), \quad (\varphi,\bar{\phi}) \mapsto \theta_\varphi(\phi).$$
- Define the global theta lift $\Theta_V(\pi) \subseteq \mathscr{A}(H(\mathbb{A}))$ of π to be its image, an $H(\mathbb{A})$-subrepresentation of $\mathscr{A}(H(\mathbb{A}))$.

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- Theory of theta correspondence provides a rather complete description of $\Theta_V(\pi)$. Key tools: Siegel–Weil formula and Rallis inner product formula.

Theta integral and Eisenstein series

Theta integral and Eisenstein series

- Associated to $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$, consider the theta integral

$$I(g, \varphi) := \int_{[H]} \theta(g, h, \varphi)\, \mathrm{d}h,$$

  When $I(g, \varphi)$ converges absolutely, it gives an automorphic form on $G(\mathbb{A})$.
- Get a $G(\mathbb{A})$-equivariant distribution

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- Associated to $\varphi \in \mathscr{S}(V(\mathbb{A})^n)$, there is a standard Siegel–Weil section $\Phi_\varphi(g,s) \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\chi|\cdot|_F^s)$:
$$\Phi_\varphi(g,s) := \omega(g)\varphi(0)\cdot|\det a(g)|_F^{s-s_0},$$
where
$$s_0 := (m-n)/2.$$
- Define the (hermitian) Siegel Eisenstein series
$$E(g,s,\varphi) := \sum_{\gamma\in P(F_0)\backslash G(F_0)} \Phi_\varphi(\gamma g,s), \quad g \in G(\mathbb{A}).$$

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- If $E(g,s,\varphi)$ is homomorphic at $s=s_0$, also get a $G(\mathbb{A})$-equivariant distribution
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Example

Take $(G, H) = (\mathrm{Sp}(2), \mathrm{O}(V))$ with $V = (\mathbb{Q}^2, x^2 + y^2)$.
Take $\varphi = \otimes\varphi_v$ with $\varphi_p = \mathbf{1}_{\mathbb{Z}_p^2}$ and $\varphi_\infty(x, y) = e^{-\pi(x^2+y^2)}$ is the Gaussian.
The Siegel–Weil formula recovers

$$\sum_{x,y\in\mathbb{Z}} q^{x^2+y^2} = 1 + 4\sum_{n\geq 1}\left(\sum_{d|n}\chi(d)\right) q^n$$

an identity of modular forms in $M_1(4,\chi)$, where $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \xrightarrow{\sim} \{\pm 1\}$. It easily implies Fermat's theorem:

A prime $p \neq 2$ is of the form $p = x^2 + y^2$ if and only if $p \equiv 1 \pmod 4$.

Doubling zeta integrals

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- $W^\square := W \oplus (-W)$ skew-hermitian space of dimension $4n$ over F.
- $G^\square := \mathrm{U}(W^\square)$, a quasi-split unitary group over F_0 in $4n$-variables.
- For $\varphi_1, \varphi_2 \in \mathscr{S}(V(\mathbb{A})^n)$, have a Siegel Eisenstein series $E(g, s, \varphi_1 \otimes \overline{\varphi_2})$ on $G^\square$.
- π cuspidal automorphic representation of $G(\mathbb{A})$.
- For $\phi_1, \phi_2 \in \pi$, define the global doubling zeta integral

$$Z(s, \phi_1, \phi_2, \varphi_1, \varphi_2) := \int_{[G]\times[G]} \overline{\phi_1}(g_1)\phi_2(g_2)\cdot E((g_1, g_2), s, \varphi_1 \otimes \overline{\varphi_2})\, \chi^{-1}(\det g_2)\, \mathrm{d}g_1 \mathrm{d}g_2.$$

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- When $\varphi_i = \otimes_v \varphi_{i,v}$ and $\phi_i = \otimes_v \phi_{i,v}$ it factors into local doubling zeta integrals

$$Z(s, \phi_1, \phi_2, \varphi_1, \varphi_2) = \prod_v Z_v(s, \phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}).$$

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- When all the data are unramified at a finite place v with $\langle \phi_{1,v}, \phi_{2,v} \rangle = 1$, we have

$$Z_v(s, \phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}) = \frac{L(s + 1/2, \pi_v \times \chi_v)}{b_{2n,v}(s)}.$$

  - $L(s + 1/2, \pi_v \times \chi_v)$ is the doubling L-factor [Harris–Kudla–Sweet, Lapid–Rallis, Yamana,...]
  - $b_{k,v}(s) := \prod_{i=1}^{k} L(2s + i, \eta_v^{k-i})$ is a product of Hecke L-factors.

Rallis inner product formula

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- Define the normalized local doubling zeta integral

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then $Z_v^\natural(s,\phi_{1,v},\phi_{2,v},\varphi_{1,v},\varphi_{2,v}) = 1$ for almost all v.

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Theorem (Rallis inner product formula, [Rallis, J.-S. Li, ...])
Assume that the pair $(V, W^\square)$ satisfies Weil's convergence condition. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Then for any $\phi_i = \otimes_v \phi_{i,v} \in \pi$, $\varphi_i = \otimes_v \varphi_{i,v} \in \mathscr{S}(V(\mathbb{A})^n)$ $(i = 1, 2)$,

$$\langle \theta_{\varphi_1}(\phi_1), \theta_{\varphi_2}(\phi_2) \rangle_H \doteq \frac{L(s_0 + 1/2, \pi \times \chi)}{b_{2n}(s_0)} \cdot \prod_v Z_v^\natural(s_0, \phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v}).$$

Here $s_0 = (m - 2n)/2$ as in the Siegel–Weil formula for the pair $(V, W^\square)$.

Theta dichotomy in the equal rank case

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- Take $m = 2n$ (two spaces V, W have equal rank) and $\chi = 1$.
- The special point $s_0 = 0$ corresponds to the center of the function equation of the Eisenstein series, and central L-values $L(1/2, \pi)$.

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- The special point $s_0 = 0$ corresponds to the center of the function equation of the Eisenstein series, and central L-values $L(1/2, \pi)$.
- By the Rallis inner product formula:

$$\text{global theta lifting } \Theta_V(\pi) \neq 0 \iff L(1/2, \pi) \neq 0, \text{ and } \prod_v Z_v^\natural(0) \neq 0.$$

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- The Birch–Swinnerton-Dyer conjecture (and generalization by Beilinson–Bloch):

$$L'(1/2, \pi) \neq 0 \overset{?}{\longleftrightarrow} \text{non-triviality of algebraic cycles}.$$

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These were conjectured by Kudla in '90s and were largely settled for quaternionic Shimura curves over $\mathbb{Q}$, cumulating in the monograph [Kudla–Rapoport–Yang 2006]. These conjectures for Shimura varieties of higher dimension were seemingly far from reach at that time, yet recent years have witnessed advances on all of them.

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- X_K is a Shimura variety of abelian type. Its étale cohomology and L-function are computed in forthcoming [Kisin–Shin–Zhu], under the help of the endoscopic classification for unitary groups [Mok, Kaletha–Minguez–Shin–White].

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- Assume $\mathbb{V}$ is totally definite: $\mathbb{V}$ has signature $(m, 0)$ at all real places.
- For any embedding $\sigma : F \hookrightarrow \mathbb{C}$, we have a unique standard indefinite hermitian space V, depending on σ, such that V_v has signature $(m-1, 1)$ at the real place of F_0 induced by σ, and $\mathbb{V}_v \simeq V_v$ at all other places of F_0.
- Theory of conjugation of Shimura varieties: the Shimura variety X_K can be intrinsically defined over F (without being viewed as a subfield of $\mathbb{C}$).
- Thus a totally definite incoherent hermitian space $\mathbb{V}$ over $\mathbb{A}_F$ gives a system of unitary Shimura varieties $X = \{X_K\}$ canonically defined over F.

Remark (Dichotomy)

- Geometric invariants of X_K (over the algebraically closed field $\mathbb{C}$), a choice of the embedding $\sigma : F \hookrightarrow \mathbb{C}$ is involved. The coherent space $V(\mathbb{A})$ associated to V should play a canonical role and special values of analytic quantities ought to appear.

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- Arithmetic invariants of X_K (over the number field F), no choice of the embedding $\sigma : F \hookrightarrow \mathbb{C}$ is involved. The incoherent space $\mathbb{V}$ should play a canonical role and special derivatives of analytic quantities ought to appear.

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  here $\cap$ denotes the fiber product over X_K.
- It only depends on the F-span $V_{\mathbf{x}}$ of $\{x_1, \ldots, x_n\}$ in $\mathbb{V}_f^n$ and we write $Z(V_{\mathbf{x}})_K := Z(\mathbf{x})_K$.

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- If $(x, x) > 0$, then the excess intersection formula implies that

$$Z(x)_K \cdot Z(x)_K = Z(x)_K \cdot c_1(\mathcal{L}_K^\vee) \in \mathrm{CH}^2(X_K).$$

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- This motivates us to define

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  which is an element in the Chow group of correct codimension.
- For a K-invariant Schwartz function $\varphi \in \mathscr{S}(\mathbb{V}_f^n)^K$ and $T \in \mathrm{Herm}_n(F_0)_{\geq 0}$, define the weighted special cycle

$$Z(T, \varphi)_K = \sum_{\substack{\mathbf{x} \in K \backslash \mathbb{V}_f^n \\ (\mathbf{x}, \mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x})_K \in \mathrm{CH}^n(X_K)_{\mathbb{C}}.$$

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- Define arithmetic theta function (or Kudla's generating function)

$$Z(\tau,\varphi)_K = \sum_{T\in \mathrm{Herm}_n(F)_{\geq 0}} Z(T,\varphi)_K \cdot q^T,$$

as a formal generating function valued in $\mathrm{CH}^n(X_K)_{\mathbb{C}}$, where

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lies in the hermitian half space and $q^T := \prod_{v|\infty} e^{2\pi i \,\mathrm{tr}\, T\tau_v}$.

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$$Z(g,\varphi)_K := \sum_{T\in \mathrm{Herm}_n(F)_{\geq 0}} Z(T,\omega_f(g_f)\varphi)_K \cdot \omega_\infty(g_\infty)\varphi_\infty(T), \quad g\in G(\mathbb{A})$$

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$$Z(g,\varphi)_K = \sum_{\mathbf{x}\in K\backslash \mathbb{V}_f^n} \omega(g)(\varphi\otimes\varphi_\infty)(\mathbf{x})\cdot Z(\mathbf{x})_K,$$
- $Z(g,\varphi)_K$ is compatible under pullback when varying $K\subseteq H(\mathbb{A}_f)$ and thus defines a formal sum $Z(g,\varphi) := (Z(g,\varphi)_K)_K$ valued in $\mathrm{CH}^n(X)_{\mathbb{C}} := \varinjlim_K \mathrm{CH}^n(X_K)_{\mathbb{C}}$.

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Remark

In fact [Kudla–Millson] proves a much more general theorem, applicable to the generating function of special cohomology classes for locally symmetric spaces associated to any $\mathrm{U}(p, q)$ or $\mathrm{O}(p, q)$.

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- The proof replies on the Kudla–Millson Schwartz forms

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  - v_0 is the real place of F_0 induced by the fixed embedding $\sigma : F \hookrightarrow \mathbb{C}$,
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  and the Kudla–Millson theta function

$$\theta_{\mathrm{KM}}(g, h, \varphi) := \sum_{\mathbf{x}\in V^n} \omega(g)\widetilde{\varphi}^V(h^{-1}\mathbf{x}), \quad g \in G(\mathbb{A}), h \in H(\mathbb{A}_f),$$

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  where $\varphi_v \in \mathscr{S}(V_v^n)$ is the Gaussian function.
- Define

$$\widetilde{\varphi}^V := \varphi \otimes \widetilde{\varphi}_\infty \in \mathscr{S}(V(\mathbb{A})^n) \otimes \Omega^{n,n}(\mathbb{D})$$

  and the Kudla–Millson theta function

$$\theta_{\mathrm{KM}}(g, h, \varphi) := \sum_{\mathbf{x}\in V^n} \omega(g)\widetilde{\varphi}^V(h^{-1}\mathbf{x}), \quad g \in G(\mathbb{A}), h \in H(\mathbb{A}_f),$$

  which gives a closed (n, n)-form on $X_K(\mathbb{C})$ at any $g \in G(\mathbb{A})$.
- By the Poisson summation, $\theta_{\mathrm{KM}}(g, h, \varphi)$ defines a (nonholomorphic) automorphic form valued in closed (n, n)-forms on $X_K(\mathbb{C})$.

Geometric modularity: proof ingredient

- The proof replies on the Kudla–Millson Schwartz forms
$$\varphi_{\mathrm{KM},v_0}\in\mathscr{S}(V_{v_0}^n)\otimes\Omega^{n,n}(\mathbb{D}),$$
  - v_0 is the real place of F_0 induced by the fixed embedding $\sigma: F\hookrightarrow\mathbb{C}$,
  - $\Omega^{a,b}(\mathbb{D})$ is the space of smooth differential forms on $\mathbb{D}$ of type (a,b).
- The Schwartz form $\varphi_{\mathrm{KM},v_0}$ is $H_{v_0}(\mathbb{R})$-invariant and closed at any $\mathbf{x}\in V_{v_0}^n$.
- Define
$$\widetilde{\varphi}_\infty=\varphi_{\mathrm{KM},v_0}\otimes\bigotimes_{v|\infty,v\neq v_0}\varphi_v\in\mathscr{S}(V_\infty^n)\otimes\Omega^{n,n}(\mathbb{D}),$$
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which gives a closed (n,n)-form on $X_K(\mathbb{C})$ at any $g\in G(\mathbb{A})$.
- By the Poisson summation, $\theta_{\mathrm{KM}}(g,h,\varphi)$ defines a (nonholomorphic) automorphic form valued in closed (n,n)-forms on $X_K(\mathbb{C})$.
- $\theta_{\mathrm{KM}}(g,h,\varphi)$ represents the (holomorphic) series $[Z(g,\varphi)_K]$ in $\mathrm{H}^{2n}(X_K(\mathbb{C}),\mathbb{C})$ (in particular, the nonholomorphic terms in $\theta_{\mathrm{KM}}(g,h,\varphi)$ are exact forms).

Summary

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Clas.	$\theta(g,h,\varphi)$	$I(g,\varphi) \doteq E(g,s_0,\varphi)$	$\langle \theta_\varphi(\phi), \theta_\varphi(\phi) \rangle_H \doteq L(s_0+\frac{1}{2},\pi)$

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Geometric theta lifting

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- Geometric modularity gives an geometric theta distribution

$$Z : \mathscr{S}(\mathbb{V}_f^n) \to \mathscr{A}(G(\mathbb{A})) \otimes \mathrm{H}^{2n}(X(\mathbb{C}), \mathbb{C}), \quad \varphi \mapsto [Z(-, \varphi)].$$

  It is $G(\mathbb{A}_f) \times H(\mathbb{A}_f)$-equivariant, where $H(\mathbb{A}_f)$ acts on $\mathrm{H}^{2n}(X(\mathbb{C}), \mathbb{C})$ via the Hecke correspondences.

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- For $\phi \in \mathscr{A}_{m/2,\chi}(G(\mathbb{A}))$, define the geometric theta lift or Kudla–Millson lift
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- Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ such that $\pi_{m/2,\chi} := \pi \cap \mathscr{A}_{m/2,\chi}(G(\mathbb{A})) \neq 0$.
- Then we obtain an $G(\mathbb{A}_f) \times H(\mathbb{A}_f)$-equivariant linear map
$$\theta^{\mathrm{KM}} : \mathscr{S}(\mathbb{V}_f^n) \otimes \pi_{m/2,\chi}^\vee \to \mathrm{H}^{2n}(X(\mathbb{C}), \mathbb{C}), \quad (\varphi, \bar{\phi}) \mapsto \theta_\varphi^{\mathrm{KM}}(\phi).$$
- Define the geometric theta lift $\Theta_V^{\mathrm{KM}}(\pi) \subseteq \mathrm{H}^{2n}(X(\mathbb{C}), \mathbb{C})$ of π to be its image.

Geometric Siegel–Weil formula

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- Define the normalized geometric volume
$$\mathrm{vol}^\natural : \mathrm{H}^{2n}(X_K(\mathbb{C}), \mathbb{C}) \to \mathbb{C}, \quad [Z] \mapsto \frac{\mathrm{vol}([Z])}{\mathrm{vol}([X_K])}.$$
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Theorem (Geometric Siegel–Weil formula [Kudla])

Let $s_0 = (m-n)/2$. For any $\varphi \in \mathscr{S}(\mathbb{V}_f^n)$,
$$\mathrm{vol}^\natural([Z(g,\varphi)] \doteq E(g, s_0, \varphi^V).$$

Here $\varphi^V \in \mathscr{S}(V(\mathbb{A})^n)$ is constructed from Schwartz form $\widetilde{\varphi}^V \in \mathscr{S}(V(\mathbb{A})^n) \otimes \Omega^{n,n}(\mathbb{D})$.

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- Assume that $2n \leq \dim X_K = m-1$. Define the geometric inner product

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Let $s_0 = (m-2n)/2$. Assume $\pi_{m/2,\chi} := \pi \cap \mathscr{A}_{m/2,\chi}(G(\mathbb{A})) \neq 0$. Then for any $\phi_i = \otimes_v \phi_{i,v} \in \pi_{m/2,\chi}$, $\varphi_i = \otimes_v \varphi_{i,v} \in \mathscr{S}(\mathbb{V}_f^n)$ $(i = 1,2)$,

$$\langle \theta^{\mathrm{KM}}_{\varphi_1}(\phi_1), \theta^{\mathrm{KM}}_{\varphi_2}(\phi_2)\rangle_{X(\mathbb{C})} \doteq \frac{L(s_0+1/2,\pi\times\chi)}{b_{2n}(s_0)} \cdot \prod_v Z_v^\natural(s_0,\phi_{1,v},\phi_{2,v},\varphi^V_{1,v},\varphi^V_{2,v}).$$

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Example

When $2n = m - 1$, $\theta^{\mathrm{KM}}_{\varphi_i}(\phi_i)$ comes from a middle dimensional cycle and $s_0 = 1/2$:

geometric intersection number of $\theta^{\mathrm{KM}}_{\varphi_i}(\phi_i)$ = near central value $L(1, \pi \times \chi)$

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Remark

Geometric theta correspondence [Kudla–Millson, Funke–Millson, ...] have many applications to the cohomology of Shimura varieties and locally symmetric spaces. For example, [Bergeron–Millson–Moeglin] proved the Hodge conjecture and the Tate conjecture for X_K, in codimension $\leq \frac{1}{3}\dim X_K$ or $\geq \frac{2}{3}\dim X_K$.

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Geo.	$[Z(g,\varphi)]$	$\mathrm{vol}^\natural[Z(g,\varphi)] \doteq E(g,s_0,\varphi^V)$	$\langle \theta_\varphi^{\mathrm{KM}}(\phi), \theta_\varphi^{\mathrm{KM}}(\phi) \rangle_{X(\mathbb{C})} \doteq L(s_0+\frac{1}{2},\pi)$

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