Geometric and arithmetic theta correspondences

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IHES 2022 Summer School on the Langlands Program
July 27, 2022

Overview

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For dual pairs $(G, H) = (\mathrm{Sp}(W), \mathrm{O}(V))$, $(\mathrm{U}(W), \mathrm{U}(V))$...

Theta Correspondence	Lift Automorphic Forms on G to	Applications
Classical	Automorphic Forms on H	Langlands functionality
Geometric (Kudla–Millson '80s)	Cohomology classes on $\mathrm{Sh}(H)$	Hodge conjecture Tate conjecture
Arithmetic (Kudla's program '90s)	Algebraic cycles on $\mathrm{Sh}(H)$	BSD conjecture Beilinson-Bloch conjecture

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Clas.	$\theta(g, h, \varphi)$	$I(g, \varphi) \doteq E(g, s_0, \varphi)$	$\langle \theta_\varphi(\phi), \theta_\varphi(\phi) \rangle_H \doteq L(s_0 + \frac{1}{2}, \pi)$
Geo.	$[Z(g, \varphi)]$	$\mathrm{vol}^\natural[Z(g, \varphi)] \doteq E(g, s_0, \varphi^V)$	$\langle \theta_\varphi^{\mathrm{KM}}(\phi), \theta_\varphi^{\mathrm{KM}}(\phi) \rangle_{X(\mathbb{C})} \doteq L(s_0 + \frac{1}{2}, \pi)$

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- For orthogonal Shimura varieties over totally real fields, Conjecture is known for $n = 1$ [Yuan–Zhang–Zhang 2009, Bruinier 2012]. For $n > 1$, the modularity follows from the convergence [Yuan–Zhang–Zhang 2009].

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d	3	4	7	11	12	16	27	$\cdots$	67	$\cdots$
$Z(d)$	$(0, -1)$	$(0, -1)$	$(0, 0)$	$(0, -1)$	$(0, 0)$	$(1, 0)$	$(-1, -1)$	$\cdots$	$(6, -15)$	$\cdots$

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Miracle. The coefficients c_d appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$,

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So

$$\sum_{d \geq 1} Z(d) \cdot q^d = \phi \cdot P \in S^+_{3/2}(4 \cdot 37) \otimes E(\mathbb{Q})_{\mathbb{C}}$$

Arithmetic theta lifting

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- Assume Kudla's modularity conjecture. It gives an arithmetic theta distribution

$$Z : \mathscr{S}(\mathbb{V}_f^n) \to \mathscr{A}(G(\mathbb{A})) \otimes \mathrm{CH}^n(X)_{\mathbb{C}}, \quad \varphi \mapsto Z(-,\varphi).$$

  It is $G(\mathbb{A}_f) \times H(\mathbb{A}_f)$-equivariant, where $H(\mathbb{A}_f)$ acts on $\mathrm{CH}^n(X)_{\mathbb{C}}$ via the Hecke correspondences.

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- Define the arithmetic theta lift $\Theta_{\mathbb{V}}(\pi) \subseteq \mathrm{CH}^n(X)_{\mathbb{C}}$ of π to be its image.

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- Elements in $\widehat{\mathrm{CH}}^n(\mathcal{X}_K)$ are represented by $(Z, (g_{Z,\sigma})_{\sigma: F \hookrightarrow \mathbb{C}})$:
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  (1) Z is codimension n cycle on $\mathcal{X}_K$.
  (2) $g_{Z,\sigma}$ is a Green current for $Z_\sigma(\mathbb{C})$.
- The problem seeks to define canonically an explicit arithmetic generating function $\widehat{\mathcal{Z}}(\tau, \varphi)$ valued in $\widehat{\mathrm{CH}}^n(\mathcal{X})_{\mathbb{C}}$ which lifts $Z(\tau, \varphi)$ under the restriction map

$$\widehat{\mathrm{CH}}^n(\mathcal{X}) \to \mathrm{CH}^n(X),$$

and such that $\widehat{\mathcal{Z}}(\tau, \varphi)$ is modular.

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- A is an abelian scheme over S.
- ι is an action of O_F on A satisfying the Kottwitz condition of signature $(m-1, 1)$,

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- $\mathcal{F}_A \subseteq \operatorname{Lie} A$ is an O_F-stable $\mathcal{O}_S$-module local direct summand of rank $m-1$, satisfying the Krämer condition: O_F acts on $\mathcal{F}_A$ via the structure morphism and acts on the line bundle $\operatorname{Lie} A/\mathcal{F}_A$ via the conjugate of the structure morphism.

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- Consider the special case $F_0 = \mathbb{Q}$ and there is a global self-dual hermitian lattice Λ over O_F such that $K \subseteq H(\mathbb{A}_f)$ is the stabilizer of $\Lambda \otimes \mathbb{A}_f$.
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Its generic fiber is the product of X_K and a 0-dimensional Shimura variety.

Special divisors in arithmetic Chow groups

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$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(0)_K = \widehat{\mathcal{L}}_K^\vee + (\mathrm{Exc}, -\log|\mathrm{disc}(F)|) \in \widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$$
  - $\widehat{\mathcal{L}}_K^\vee$ is the metrized dual tautological line bundle over $\mathcal{X}_K^*$.
  - Exc is an effective vertical divisor supported above ramified places, equipped with the constant Green function $-\log|\mathrm{disc}(F)|$.

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Define the generating function in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$

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- One can also use Kudla's Green function (depending on a parameter $y = \mathrm{Im}(\tau) \in \mathbb{R}_{>0}$) in place of the automorphic Green function to obtain a nonholomorphic modular form. This is a consequence of Theorem and the modularity of the difference of the two generating functions [Ehlen–Sankaran].

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(1) Theorem allows one to construct arithmetic theta lifts valued in $\widehat{\mathrm{CH}}^1(\mathcal{X}_K^*)$. [BHKRY] prove formulas relating the arithmetic intersection of these arithmetic theta lifts and small/big CM points to the central derivative of certain convolution L-functions of two elliptic modular forms.

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(5) The proof of the averaged Colmez conjecture in [Andreatta–Goren–Howard–Madapusi Pera] relies on relating arithmetic intersection of special divisors on orthogonal Shimura varieties and big CM points to central derivatives of certain L-functions.

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- It is also subtle to find the correction terms at places of bad reduction and at boundary (both issues already appear when $n = 1$) and to find the correct construction of Green currents to ensure modularity.
- The forthcoming works of Howard–Madapusi Pera and Madapusi Pera address some of these issues when $n > 1$.

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- If the arithmetic theta function $\widehat{\mathcal{Z}}(\tau, \varphi) \in \widehat{\mathrm{CH}}^n(\mathcal{X}_K)_{\mathbb{C}}$ can be constructed, then we may apply the arithmetic volume

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- However, the definition of $\widehat{\mathcal{Z}}(\tau,\varphi)$ is rather subtle when $n > 1$.

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- Even for nonsingular terms, the relation to Siegel Eisenstein series is more complicated due to places of bad reduction, a phenomenon first discovered by [Kudla–Rapoport] via explicit computation in the context of Shimura curves uniformized by the Drinfeld p-adic half plane.

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- $\mathcal{X}_K$ is smooth above inert places and semistable above ramified places.
- When $F_0 = \mathbb{Q}$ and K is the stabilizer of a self-dual hermitian lattice, the regular integral model $\mathcal{X}_K$ recovers what we defined before.

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- For $v \mid \infty$, Using the Green current given by the star product of Kudla's Green functions, also define its local arithmetic intersection number $\mathrm{Int}_{T,v}(\mathrm{y}, \varphi)$ at infinite places, which depends on an additional parameter $\mathrm{y} = \mathrm{im}(\tau) \in \mathrm{Herm}_n(F_{0,\infty})_{>0}$.

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$$\widehat{\deg}_T(\mathrm{y}, \varphi) := \frac{1}{\mathrm{vol}([X_K])} \left(\sum_{v \nmid \infty} \mathrm{Int}_{T,v}(\varphi) + \sum_{v \mid \infty} \mathrm{Int}_{T,v}(\mathrm{y}, \varphi) \right).$$

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- Form the generating function of arithmetic degrees

$$\widehat{\deg}(\tau, \varphi) := \sum_{\substack{T \in \mathrm{Herm}_n(F_0) \\ \det T \neq 0}} \widehat{\deg}_T(\mathsf{y}, \varphi) q^T.$$

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- Over function fields, [Feng–Yun–Zhang 2021a] proved a higher Siegel–Weil formula for unitary groups in the unramified setting. [Feng–Yun–Zhang 2021b] further formulated the conjectural singular terms identity using both classical and derived algebraic geometry.

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Assumptions. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$.

(1) For $v|\infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{\frac{m-1}{2}, \frac{m-3}{2}, \ldots, \frac{-m+3}{2}, \frac{-m+1}{2}\}$.

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Theorem (Arithmetic inner product formula [L.–Liu, 2020,2021])

Assume Assumptions. Assume that $\varepsilon(\pi) = -1$. Assume that Kudla's modularity holds. Then for any $\phi_i = \otimes_v \phi_{i,v} \in \pi \cap \mathscr{A}_n(G(\mathbb{A}))$, $\varphi_i = \otimes_v \varphi_{i,v} \in \mathscr{S}(\mathbb{V}_f^n)$ $(i = 1, 2)$,

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In particular,

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and the converse also holds if $\langle\ ,\ \rangle_X$ is nondegenerate.

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Compatible with our formula!

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$$\text{central order of vanishing of } L_p(\pi) \text{ is } 1 \Longrightarrow \operatorname{rank} \mathrm{H}^1_f(F, \rho_\pi(n)) \geq 1.$$

This verifies part of the *p*-adic Bloch–Kato conjecture.

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$$\text{central order of vanishing of } L_p(\pi) \text{ is } 1 \Longrightarrow \mathrm{rank}\, \mathrm{H}^1_f(F, \rho_\pi(n)) \geq 1.$$

This verifies part of the p-adic Bloch–Kato conjecture.

Remark

[Xue 2019] used the arithmetic inner product formula in the case $n = 1$ to prove endoscopic cases of the arithmetic GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(3)$.

Application to the Beilinson–Bloch conjecture

Theorem (L.–Liu 2020, 2021)

Assume Assumptions. Let $\mathrm{CH}^n(X)^0_{\mathfrak{m}_\pi}$ the localization of $\mathrm{CH}^n(X)^0_{\mathbb{C}}$ at the maximal ideal $\mathfrak{m}_\pi$ of the spherical Hecke algebra of $H(\mathbb{A}_f)$ (away from all ramification) associated to π. Then the implication

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \Longrightarrow \operatorname{rank} \mathrm{CH}^n(X)^0_{\mathfrak{m}_\pi} \geq 1$$

holds when the level subgroup $K \subseteq H(\mathbb{A}_f)$ is sufficiently small.

Remark

[Disegni–Liu 2022] proved a *p*-adic arithmetic inner product formula,

$$p\text{-adic height pairing of } \Theta_\varphi(\phi) \doteq \text{derivative of cyclotomic } p\text{-adic } L\text{-function } L_p(\pi)$$

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Remark

[Xue 2019] used the arithmetic inner product formula in the case $n = 1$ to prove endoscopic cases of the arithmetic GGP conjecture for $\mathrm{U}(2) \times \mathrm{U}(3)$.
One also expects similar applications to endoscopic cases of arithmetic GGP conjecture for $\mathrm{U}(m) \times \mathrm{U}(m+1)$.

Summary

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	Theta	Siegel–Weil formula	Inner product formula
Clas.	$\theta(g,h,\varphi)$	$I(g,\varphi) \doteq E(g,s_0,\varphi)$	$\langle \theta_\varphi(\phi), \theta_\varphi(\phi)\rangle_H \doteq L(s_0+\frac{1}{2},\pi)$
Geo.	$[Z(g,\varphi)]$	$\mathrm{vol}^\natural[Z(g,\varphi)] \doteq E(g,s_0,\varphi^V)$	$\langle \theta_\varphi^{\mathrm{KM}}(\phi), \theta_\varphi^{\mathrm{KM}}(\phi)\rangle_{X(\mathbb{C})} \doteq L(s_0+\frac{1}{2},\pi)$
Ari.	$Z(g,\varphi)$	?	?

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Geo.	$[Z(g, \varphi)]$	$\mathrm{vol}^\natural [Z(g, \varphi)] \doteq E(g, s_0, \varphi^V)$	$\langle \theta_\varphi^{\mathrm{KM}}(\phi), \theta_\varphi^{\mathrm{KM}}(\phi) \rangle_{X(\mathbb{C})} \doteq L(s_0 + \frac{1}{2}, \pi)$
Ari.	$Z(g, \varphi)$	$\widehat{\deg}(\tau, \varphi) \doteq E'(\tau, 0, \varphi^{\mathbb{V}}) + \ldots$	?

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	Theta	Siegel–Weil formula	Inner product formula
Clas.	$\theta(g,h,\varphi)$	$I(g,\varphi) \doteq E(g,s_0,\varphi)$	$\langle \theta_\varphi(\phi), \theta_\varphi(\phi)\rangle_H \doteq L(s_0+\frac{1}{2},\pi)$
Geo.	$[Z(g,\varphi)]$	$\mathrm{vol}^\natural[Z(g,\varphi)] \doteq E(g,s_0,\varphi^V)$	$\langle \theta_\varphi^{\mathrm{KM}}(\phi), \theta_\varphi^{\mathrm{KM}}(\phi)\rangle_{X(\mathbb{C})} \doteq L(s_0+\frac{1}{2},\pi)$
Ari.	$Z(g,\varphi)$	$\widehat{\deg}(\tau,\varphi) \doteq E'(\tau,0,\varphi^{\mathbb{V}}) + \ldots$	$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi)\rangle_X \doteq L'(\frac{1}{2},\pi)$