2022 Summer School on the Langlands Program at IHES

SHIMURA VARIETIES: OUTLINE (1/3)

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(NOTES BY WENHAN DAI)

Abstract. Depending on your point of view, Shimura varieties are a special kind of locally symmetric spaces, a generalization of moduli spaces of abelian schemes with extra structures, or the imperfect characteristic 0 version of moduli spaces of shtuka. They play an important role in the Langlands program because they have many symmetries (the Hecke correspondences) allowing us to link their cohomology to the theory of automorphic representations, and on the other hand they are explicit enough for this cohomology to be computable.

The goal of these lectures is to give an introduction to Shimura varieties, to present some examples, and to explain the conjectures on their cohomology (at least in the simplest case).

1. Motivation: Matsushima's formula

In a non-historical sense, **Matsushima's formula** is one of many reasonable aspects to understand the motivation to care about Shimura varieties. Matsushima proved some results about the homology of the following objects in the case where they are compact.

1.1. Basic notions.

- **Let G be a connected reductive group over $\mathbb{Q}$.**
  Caution: you might prefer to image $G = \mathrm{GL}_n$ but this is not going to work so well for Shimura varieties as we will discover in a short while. Yet you can still think about $G = \mathrm{GL}_2$ but not $G = \mathrm{GL}_3$ for some reason.
- **Denote $K_\infty \subset G(\mathbb{R})$ the maximal compact subgroup modulo center. Then there is a symmetric space $X = G(\mathbb{R})/K_\infty$, which is a nice Riemann manifold.** In this case, we can pretend G to be semisimple. But in general, we do not need this assumption anymore. Morally, there will be a canonical way to choose K_∞.
- **Suppose $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup which is torsion-free. Hence there is a locally symmetric space $\Gamma\backslash X$.**
  The arithmeticity loosely means some similarity to $G(\mathbb{Z})$. However, note that $G(\mathbb{Z})$ cannot be well-defined because it depends on an embedding $G \hookrightarrow \mathrm{GL}_n$ for some sufficiently large n. Actually, Γ here is commensurable with some choice of $G(\mathbb{Z})$ that does not depend on the choice. The torsion-freeness is necessary; note that Γ is automatically torsion-free if it is small enough.

The problem is to discover when these objects are compact. In fact,

- ◇ $\Gamma\backslash X$ is compact $\iff$ $G/Z(G)$ is anisotropic over $\mathbb{Q}$ (i.e., $G/Z(G)$ have no proper parabolic subgroup).

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Do not be confused because we are not assuming that G is semisimple. Alternatively, if G is actually semisimple, it is the same as saying that $G(\mathbb{R})$ (or Γ) is compact. Simply in the reductive case, we must modulo the center $Z(G)$.

Example 1.1. Here we do not want $G(\mathbb{R})$ to be anisotropic, but we want G to be anisotropic over $\mathbb{Q}$. For a nontrivial example, we could take

$$G = D^\times,$$

where $D/\mathbb{Q}$ is a division algebra.

1.2. **An adelic reformulation of Matsushima's formula.** The work of Matsushima[1] related the Betti numbers, say $\dim H^i(\Gamma\backslash X)$, to automorphic forms on $G(\mathbb{R})$ when $\Gamma\backslash X$ is compact. But he only worked with some semisimple groups.

In the present context, we are to introduce the stuffs that are going to appear in Matsushima's formula.

(1) Recall that the **Adeles** over $\mathbb{Q}$ is the restricted product of all completions with respect to all places of $\mathbb{Q}$, i.e.,

$$\mathbb{A} = \prod_v{}' \mathbb{Q}_v = \mathbb{A}_f \times \mathbb{R},$$

breaking into the finite and the infinite parts. Moreover, the **finite Adeles**

$$\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is a locally compact topological ring.

(2) Fix a Haar measure on G. Now we introduce a nice Hilbert space:

$$\underset{\substack{\circlearrowleft \\ G(\mathbb{A})}}{\boxed{L^2_G}} := L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/A_G)$$

equipped with an action of $G(\mathbb{A})$ given by right translations. The only ambiguity lies in the description of A_G: if $S_G = Z(G)^{\text{split}}$ is the split center of G, then $A_G := S_G(\mathbb{R})^\circ$ is just the connected component in $S_G(\mathbb{R})$. (The upshot for this construction is to make sure that the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})/A_G$ has a finite volume.

(3) We now concern about a set of discrete automorphic representations of $G(\mathbb{A})$. This is, by definition,

$$\Pi(G) := \{\text{irreducible representations } \pi \text{ of } G(\mathbb{A}) \text{ that are direct factors of } L^2_G\}$$

Also, for each $\pi \in \Pi(G)$, we define

$$m(\pi) := \text{the multiplicity of } \pi \text{ while appearing in } L^2_G.$$

The factorization $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ renders that $G(\mathbb{A}) = G(\mathbb{A}_f) \times G(\mathbb{R})$, and hence leads to two representations, say π_f and π_∞, respectively. Then for each $\pi \in \Pi(G)$, which is an irreducible automorphic representation of $G(\mathbb{A})$,

$$\pi = \pi_f \otimes \pi_\infty.$$

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[1] This is not the original result by Matsushima but a reformulation instead. More precisely, now we are doing automorphic forms on $G(\mathbb{A})$ instead of $G(\mathbb{R})$ so one may need to talk about the identical formulation about this.

(4) Assume $G/Z(G)$ is anisotropic. Then, by the previous fact,

$$G(\mathbb{Q})\backslash G(\mathbb{A})/A_G \text{ is compact.}$$

And we obtain a factorization

$$L^2_G = \widehat{\bigoplus_{\pi\in\Pi(G)}} \pi^{m(\pi)}.$$

All these automorphic representations in $\Pi(G)$ are also capital in the course (because there are no parabolics).

(5) There is another slight adelic variance of the space, which is called the **adelic locally symmetric spaces**,

$$M_K := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K),$$

where K is an open compact subgroup of $G(\mathbb{A}_f)$ over finite adelic points. This space M_K looks like a different beast for those who have never seen it before, but actually, it is basically the same object as $\Gamma\backslash X$. In the double coset, K acts by right translation on $G(\mathbb{A}_f)$ and $G(\mathbb{Q})$ acts on both factors simultaneously by left multiplication (and hence it acts diagonally).

Remark. If we write $G(\mathbb{A}_f)$ as a finite union of double classes like

$$G(\mathbb{A}_f) = \bigsqcup_{i\in I} G(\mathbb{Q})x_iK,$$

then

$$M_K = \bigsqcup_{i\in I} \Gamma_i\backslash X, \quad \Gamma_i = x_iKx_i^{-1} \cap G(\mathbb{Q}).$$

So the open compact subgroup of $G(\mathbb{A}_f)$ is like the adelic version of arithmetic subgroup of $G(\mathbb{Q})$ (for which we want it to be a little bit more restricted). In case when K is small enough, all these Γ_i are going to be torsion-free. However,

$$\begin{array}{l} \varprojlim_K M_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f). \\ \quad\circlearrowleft \\ G(\mathbb{A}_f) \end{array}$$

We are interested in the Betti cohomology with $\mathbb{C}$-coefficients, say

$$H^i := \varinjlim_K H^i(M_K) = \varinjlim_K H^i_{\mathrm{Betti}}(M_K, \mathbb{C})$$

which is again equipped with a $G(\mathbb{A}_f)$-action. Here comes some comment on the $G(\mathbb{A}_f)$-actions before stating Matsushima's formula. Think about these actions at finite levels (namely, without taking injective limits). For some fixed open compact K_0, the injective limit H^i is the K_0-fixed points in $H^i(M_{K_0})$ – the action is not given by $G(\mathbb{A}_f)$ but by something in the Hecke algebra of level K instead[2].

(6) Denote $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$. Due to some quite strong condition at ∞, we can select some infinite part π_∞ of a fixed automorphic representation π satisfying that it has a nontrivial $(\mathfrak{g}, K)$-cohomology. In Lie theory, there is a *computable* invariant, called the **relative**

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[2] See the notes for the previous talk. More precisely, this is the algebra of locally π_K-invariant functions with compact supports on $G(\mathbb{A}_f)$.

Lie algebra cohomology that will be used in the formula (yet we choose to omit the explanation).

Theorem 1.2 (Matsushima)**.** *Assume G is anisotropic modulo the center, i.e., $G/Z(G)$ is anisotropic. As a $G(\mathbb{A}_f)$-representation,*

$$H^i \simeq \bigoplus_{\pi\in\Pi(G)} \pi_f \otimes H^i(\mathfrak{g}, K_\infty; \pi_\infty)^{m(\pi)}.$$

In short, the formula dictates two key-points as follows.

- ◇ The only representations of $G(\mathbb{A}_f)$ that appear in H^i are the finite parts of automorphic representations π_f say.
- ◇ We obtain an explicit way to calculate their multiplicities $m(\pi)$. These multiplicities are given by the product of multiplicities of π in L^2_G.

Alternatively, there is a finite-level version for Theorem 1.2.

Theorem 1.3 (Finite-level Matsushima)**.** *As $\mathscr{H}_K := \mathcal{C}_c(K\backslash G(\mathbb{A}_f)/K, \mathbb{C})$-modules*[3],

$$H^i(M_K) \simeq \bigoplus_{\pi\in\Pi(G)} \pi_f^K \otimes H^i(\mathfrak{g}, K_\infty; \pi_\infty)^{m(\pi)}$$

where π_f^K is the K-invariant factors in π_f.

Note that the right hand side of Theorem 1.3 is the same as that of Theorem 1.2. But we have to use another different cohomology theory for the left hand side.

2. Locally symmetric spaces as complex algebraic varieties

2.1. **Story of modular curves on** GL_2**.** Set $G = \mathrm{GL}_2$. Note that GL_2 is not anisotropic group but as we have remarked above, there is a generalization of Matsushima's formula to GL_2. Then for any fixed open compact K,

$$\begin{aligned} M_K &= \text{moduli curve} \\ &= \text{moduli space of elliptic curves plus some extra structures.} \end{aligned}$$

Then the M_K is exactly the set of $\mathbb{C}$-points of an algebraic variety defined over $\mathbb{Q}$. (Whereas M_K is not necessarily an algebraic variety over $\mathbb{C}$.) Hence the Betti cohomology

$$H^i(M_K) \circlearrowleft \mathrm{Gal}_{\mathbb{Q}}$$

admits an absolute Galois action of $\mathrm{Gal}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By fixing an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$,

$$\varinjlim_K H^i_{\mathrm{Betti}}(M_K, \mathbb{C}) \simeq \varinjlim_K \underset{\substack{\circlearrowleft \\ G(\mathbb{A}_f)\times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}}{\boxed{H^i_{\text{ét}}(M_{K,\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)}}.$$

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[3] Here we are working with a cocompact assumption. It turns out that we are safe to concentrate only on the cocompact case due to Bill Casselman's generalization for this to non-cocompact locally symmetric spaces, under a condition that is going to be satisfied for Shimura varieties (which is basically the fact that $G(\mathbb{R})$ has a discrete series). Later, we will see examples that are not always cocompact, so everything generalizes in the appropriate way.

There is a $G(\mathbb{A}_f)$-action for the étale cohomology for the following reason. One can basically descend all these M_K's to algebraic varieties over $\mathbb{Q}$. Then the actions of $G(\mathbb{A}_f)$ is given by the Hecke correspondence[4]. Also, because the actions of $G(\mathbb{A}_f)$ are all defined over $\mathbb{Q}$ (or locally, defined over $\mathbb{Q}_p$'s except for $\mathbb{Q}_\infty$), they commute with the $\mathrm{Gal}_{\mathbb{Q}}$-actions.

If we have an automorphic representation π that comes from a modular form, then we take a 2-dimensional Galois representation of $\mathrm{Gal}_{\mathbb{Q}}$ (à la Deligne):

$$\sigma(\pi) := H^1[\pi_f],$$

where $[\pi_f]$ denotes the π_f-isotypic component. Via the global Langlands correspondence, it turns out to be the representation $\mathrm{GL}(\pi)$ that corresponds to π. If we restrict $\sigma(\pi)$ to some local Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the result would correspond to the component of π at the place p (finite or infinite), very neatly via the local Langlands correspondence.

Upshot. From a geometric aspect, we summarize the known information for GL_2.

- ◇ By Matsushima's formula, the automorphic representations appear in the Betti cohomology of adelic locally symmetric spaces M_K's, and hence in the étale cohomologies via the isomorphism given by the Galois action.
- ◇ If the geometric stuff M_K happens to be an algebraic variety over $\mathbb{C}$, then we will see no automorphic representations in the Betti cohomology but something looks like the global Langlands correspondence (normalizing in an appropriate way) instead.

Keynote Questions. The condition goes very nice when $G = \mathrm{GL}_2$. Is there a more general picture for other reductive groups? More precisely, we have two questions.

(a) When is $\Gamma\backslash X$ (or M_K) an algebraic variety over $\mathbb{C}$?
Spoiler: the answer will be "when M_K is a Shimura variety".

(b) When it is, can we descend it to $\mathbb{Q}$ or another number field?
There's another subtlety in (b). A priori even if the algebraic variety is defined over a number field, there could still be several ways to define it. Then a nice way for definition is in need. In particular, as in the GL_2 case, we would like the action of $G(\mathbb{A}_f)$ to also descend to the number field.

2.2. Hermitian symmetric domains.

This part is to answer Question (a) above. Recall our statement: choose a maximal compact subgroup K_∞ and define the symmetric space $X = G(\mathbb{R})/K_\infty$. X is equipped with a Hermitian metric.

Definition 2.1. We say that X is a **Hermitian symmetric domain** (HSD) if X has a complex structure and a Hermitian metric such that $G(\mathbb{R})$ acts by holomorphic isometries.

Note that a Hermitian metric is just the analogue of a Riemannian metric. But now, the tangent spaces of X are vector spaces so we have to give a scalar product on each of them that varies holomorphically.

Example 2.2 (Siegel upper-half space)**.** This can be view as a higher-dimensional Poincaré upper-half plane. Let $d \geqslant 1$ be a positive integer. Take

$$\mathfrak{h}_d^+ = \{X \in \mathrm{M}_d(\mathbb{C}) \mid X^t = X,\ \mathrm{Im}(X) > 0\}$$

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[4] The Hecke correspondence automatically algebraizes if M_K is a Shimura variety.

to be the set of those $d \times d$ complex symmetric matrices whose imaginary part are positive definite. It can be checked that $\mathfrak{h}_1^+ = \mathbb{H}$, the upper-half plane. Also, there is a symplectic action

$$\mathfrak{h}_d^+ \curvearrowleft \mathrm{Sp}_{2d}(\mathbb{R}), \qquad \text{where } \mathrm{Sp}_{2d} = \left\{ g \in \mathrm{GL}_{2d} \;\middle|\; g^t \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \right\}.$$

Then Sp_{2d} is a group scheme over $\mathbb{Z}$. Usually we choose the anti-diagonal symplectic form because it is easier to write the parabolic subgroups down. But we do not concern about compactifications here (so it does not matter). One can check that $\mathrm{Sp}_2 = \mathrm{SL}_2$. The explicit action is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} X = (AX+B)(CX+D)^{-1}, \quad A, B, C, D \in \mathrm{M}_d(\mathbb{C}).$$

As an exercise, you are supposed to check the transitivity of this action.

Recall that in the $d = 1$ case, we take the fundamental domain to be $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i)$. Here the maximal compact subgroup can be chosen as

$$K_\infty = \mathrm{Stab}_{\mathrm{Sp}_{2d}(\mathbb{R})}(iI_d) = \mathrm{O}_{2d} \cap \mathrm{Sp}_{2d}(\mathbb{R}).$$

Then as a real manifold,

$$\mathfrak{h}_d^+ \simeq \mathrm{Sp}_{2d}(\mathbb{R})/K_\infty.$$

It has a complex structure and $\mathrm{Sp}_{2d}(\mathbb{R})$ does act by holomorphic maps.

Bounded Realization. Up to the scalar by a positive number, every bounded domain in a finite-dimensional $\mathbb{C}$-vector space has a canonical Bergman metric. The problem now lies in that $\mathfrak{h}_d^+$ is an unbounded domain, and the *bounded realization progress* is intended to realize it as a bounded domain in the same $\mathbb{C}$-vector space.

We claim that there is a conformal (i.e., holomorphic and bijective) equivalence

$$\begin{aligned} \mathfrak{h}_d^+ &\xrightarrow{\sim} \mathcal{D}_d := \{X \in \mathrm{M}_d(\mathbb{C}) \mid X^t = X,\ I_d X^* X > 0\} \\ X &\longmapsto (iI_d - X)(iI_d + X)^{-1}. \end{aligned}$$

Note that $\mathcal{D}_d$ (the higher-dimensional version of the unit disc) is again an open subset of the finite-dimensional $\mathbb{C}$-vector space of symmetric matrices. But this time it is bounded. Indeed, when $d = 1$, we have $\mathcal{D}_1 = \mathbb{D}$ and the map is the conformal equivalence from $\mathbb{H}$ to $\mathbb{D}$.

2.3. **Classification of Hermitian symmetric domains in terms of real groups.**

Theorem 2.3. *Suppose that $G(\mathbb{R})$ is connected adjoint*[5]*. Then X is a Hermitian symmetric domain if and only if there exists a morphism of Lie groups*

$$u : \mathrm{U}(1) \to G_{\mathrm{ad}}(\mathbb{R})$$

satisfying the following conditions[6]*.*

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[5] One can always reduce the situation to this case. On this assumption, $G(\mathbb{R})$ is going to be the group of diffeomorphisms of X.

[6] In some references, there seems to be a third condition saying that $u(i)$ is not trivial on any factor of $G(\mathbb{R})$. But this can be implied by (a) and (b).

(a) *The only characters of* $\mathrm{U}(1)$ *that appear in the algebra* $\mathrm{Lie}\, G(\mathbb{C})$*, whose actions are given by* $\mathrm{Ad} \circ u$*, are* 1*,* z*, and* z^{-1}*.*

(b) $\mathrm{Int}(u(i))$ *is a Cartan involution of* $G(\mathbb{R})$*. That is,*

$$\{g \in G(\mathbb{C}) \mid g = u(i)\overline{g}u(i)^{-1} \textit{is compact}\}$$

Moreover, we can choose some u *such that*

$$K_\infty = \mathrm{Cent}_{G(\mathbb{R})}(u),$$

the centralizer of u *in* $G(\mathbb{R})$*. Hence one may always assume* K_∞ *having this form. So*

$$X = G(\mathbb{R})/K_\infty = \{G(\mathbb{R})\textit{-conjugacy classes of } u\}.$$

Example 2.4. Since Sp_{2d} is not adjoint, we consider $G = \mathrm{PSp}_{2d}$, which shares the same symmetric space as Sp_{2d} does. Take

$$u : \mathrm{U}(1) \to \mathrm{PSp}_{2d}(\mathbb{R}), \quad a + ib \mapsto \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

We point out that the condition in Theorem 2.3 is very strong for $G(\mathbb{R})$. For example, you can check that there is no Cartan involution for $G = \mathrm{PGL}_n$ with $n \geqslant 3$.

Remark 2.5. The $G(\mathbb{R})$ having such a u are all classified by the classification of real simple Lie groups[7]. More precisely, the isomorphism classes of irreducible HSDs are classified by the spacial nodes on connected Dynkin diagrams (see [Mil17]). For example,

- as for the factors of type A of $G(\mathbb{R})$, we cannot have GL_n in this situation but can only have unitary groups $\mathrm{PSU}(p, q)$;
- there are no irreducible HSDs of type G_2, F_4, or E_8.

2.4. **The structure of algebraic variety on locally symmetric spaces.** We then introduce a big theorem of Baily and Borel (about the compactification) to finish the answer to Question (a).

Theorem 2.6 (Baily–Borel, Borel)**.** *If* X *is a Hermitian symmetric domain, then for an arbitrary arithmetic subgroup* Γ *of* $G(\mathbb{Q})$*, there is, on* $\Gamma\backslash X$*, a unique structure of quasi-projective algebraic variety over* $\mathbb{C}$*.*

Some (historical and philosophical) explanations for this theorem.

- What Baily–Borel did is that they constructed an open embedding of $\Gamma\backslash X$ into a projective algebraic variety called the Baily–Borel compactification or minimal Satake compactification.
- The upshot of Borel's exclusive proof is that if one has an algebraic variety Y over $\mathbb{C}$, then any holomorphic map $Y \to \Gamma\backslash X$ is automatically an algebraic and rational map. Borel did this by assuming Y is smooth but the resolution of singularities works well.
- In particular, the structure is unique. This also implies that our Hecke correspondence is algebraic.

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[7] See Milne's notes *Introduction to Shimura varieties*, page 20.

Remark 2.7. In Theorem 2.6 of Baily and Borel, we take no condition for Γ. If Γ is torsion-free, then $\Gamma\backslash X$ is smooth (by natural means). But if not, then $\Gamma\backslash X$ is an algebraic variety with some pretty mild singularities. This is because one can always take a smaller torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index to get a smooth quasi-projective algebraic variety $\Gamma'\backslash X$, and then $\Gamma\backslash X$ is a quotient of $\Gamma'\backslash X$ by a finite group.

In the context of adelic version, for every open compact subgroup K of $G(\mathbb{A}_f)$, the M_K has a unique structure of complex algebraic variety and it is quasi-projective.

3. Siegel Modular Varieties

The upcoming task is to descend the algebraic variety over $\mathbb{C}$ to those over number fields. So we concern about which ones admit the descent, and how shall we do for this. The example of Siegle modular varieties would guide us to the answer (by [DMK94]).

Loosely speaking, Siegle modular varieties are the higher-dimensional generalizations of modular curves. In the case of GL_2, as the story goes, say what happens with modular curves is the Siegel upper-half plane basically parametrizes elliptic curves obviously. We are to do the similar thing for parametrizing abelian varieties (as higher-dimensional elliptic curves).

The reference of the following discussions is [Mum08]. Fix an integer $d \geqslant 2$. Let A be an abelian variety[8] over $\mathbb{C}$ of dimension d. By abuse of notation for convenience, we identify A with its $\mathbb{C}$-points $A(\mathbb{C})$. Then

$$\mathrm{Lie}(A) \simeq \mathbb{C}^d, \quad A \simeq \mathrm{Lie}(A)/\Lambda$$

for some lattice

$$\Lambda \simeq \pi_1(A) \simeq H_1(A, \mathbb{Z}),$$

which is a free $\mathbb{Z}$-module of rank $2d$. The question is that not all quotients of $\mathbb{C}^d/\Lambda$ in the form of complex tori are going to be algebraic as abelian varieties. In fact, a complex torus is isomorphic to some abelian variety if and only if it is polarizable[9]. Consider

$$A^\vee = \mathrm{Lie}(A^\vee)/\Lambda^\vee,$$

where by definitions,

$$\begin{aligned} \mathrm{Lie}(A^\vee) &:= \{\text{semilinear forms } l \text{ on } \mathrm{Lie}(A)\}, \\ \Lambda^\vee &:= \{l \in \mathrm{Lie}(A^\vee) \mid \mathrm{Im}\, l(\Lambda) \subset \mathbb{Z}\}. \end{aligned}$$

Proposition 3.1. *We have the following bijection:*

$$\left\{\textit{polarizations } \lambda : A \to A^\vee\right\} \longleftrightarrow \left\{\begin{matrix} \textit{positive definite Hermitian forms } H \\ \textit{on } \mathrm{Lie}(A) \textit{ such that } \mathrm{Im}\, H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{matrix}\right\}.$$

$$\begin{array}{rcl} \lambda & \longmapsto & H_\lambda \\ \lambda_H : v \mapsto H(v, \cdot) & \longleftarrow\!\shortmid & H \end{array}$$

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[8] Recall that an abelian variety over $\mathbb{C}$ is a connected compact complex Lie group. Then the exponential map $\exp : \mathrm{Lie}(A) \to A$ induces the universal cover.

[9] Equivalently, say there is a Riemann form on $\mathbb{C}^d/\Lambda$. The Riemann form is some positive definite Hermitian form satisfying some further conditions.

It turns out that each polarization of A is in the form of λ_H for some H. The Weil pairing is induced from this, say

$$\begin{tikzcd} A[n] \times A[n] \arrow[r, "\mathrm{id}\times\lambda_H"] \arrow[rr, bend left, "\psi_{\lambda_H}"] & A[n] \times A^\vee[n] \arrow[r] & \mu_n(\mathbb{C}) \\ (v,w) \arrow[rr] & & \exp(-2\pi i n H(v,w)) \end{tikzcd}$$

where $A[n] := \{P \in A \mid [n]P = O\}$ denote the n-torsion points (and similarly for $A^\vee[n]$).

Proposition 3.2. *At the level of sets, the upper-half space is in bijection with the set of equivalence classes of the following triples:*

$$\mathfrak{h}_d^+ \longleftrightarrow \{(A, \lambda, \eta_{\mathbb{Z}})\}/\sim .$$

Here

- *A is an abelian variety over $\mathbb{C}$ of dimension d;*
- *$\lambda : A \xrightarrow{\sim} A^\vee$ is a principal polarization (i.e., a polarization that is an isomorphism);*
- *$\eta_{\mathbb{Z}} : H_1(A, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^{2d}$ is a symplectic isomorphism*[10] *(called the level structure);*
- *two triples $(A, \lambda, \eta_{\mathbb{Z}}) \sim (A', \lambda', \eta'_{\mathbb{Z}})$ are equivalent if there is a quasi-isogeny $\alpha : A \to A'$ between abelian varieties that are compatible with the polarization and the level structure; that is, α satisfies $\eta'_{\mathbb{Z}} = \alpha_* \circ \eta_{\mathbb{Z}}$ and $\lambda' = \alpha^\vee \circ \lambda \circ \alpha$.*

Proof. It suffices to write down the map explicitly. For each $\tau \in \mathfrak{h}_d^+$, we define its image by the triple $(A_\tau, \lambda_\tau, \eta_{\tau,\mathbb{Z}})$. Here

- $A_\tau = \mathbb{C}^d/(\mathbb{Z}^d + \tau\mathbb{Z}^d)$ is the higher-dimensional complex torus;
- $\lambda_\tau = \lambda_{H_\tau}$, where $H_\tau = (\operatorname{Im}\tau)^{-1}$ (recall that the imaginary part of τ is always a positive definite matrix);
- $\eta_{\tau,\mathbb{Z}} : H_1(A, \mathbb{Z}) = \mathbb{Z}^d + \tau\mathbb{Z}^d \to \mathbb{Z}^{2d}$ such that for all $1 \leqslant i \leqslant d$,

$$\eta_{\tau,\mathbb{Z}} : e_i \mapsto e_i, \quad \tau e_i \mapsto e_i + d.$$

It can be check that A_τ on this construction is truly an abelian variety because it obtains a polarization λ_τ and is isomorphic to a complex torus. The trick for proof is to use, for example, the Kodaira embedding theorem, to deduce that the polarization does define the projective embedding of A_τ. □

To be continued in Lecture 2/3.

References

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[10] Note that we have a symplectic form given by the polarization together with the standard symplectic form with the same matrix $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ as we have used before.

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