Shimura Varieties (1/3)

Sophie Morel — July 11

§1 Matsushima's formula

(a motivation to care about Shimura varieties).

G connected reductive $/\mathbb{Q}$.

$K_\infty \subseteq G(\mathbb{R})$ max'l compact modulo center.

$X = G(\mathbb{R})/K_\infty$ symmetric space (a nice Riemann manifold).

$\Gamma \subseteq G(\mathbb{Q})$ arithmetic subgroup, (may assume: torsion-free).

$\Gamma \backslash X$ locally symmetric space.

<u>Fact</u> $\Gamma\backslash X$ compact $\Leftrightarrow$ $G/Z(G)$ anisotropic over $\mathbb{Q}$.
(not only for semisimple G's).

E.g. Take $G = D^\times$, $D/\mathbb{Q}$ division algebra.

- Matsushima (reformulated ver.):
  Relates $\dim H^i(\Gamma\backslash X)$ to automorphic forms on $G(\mathbb{R})$.
  (When $\Gamma\backslash X$ is compact.)
- Adelic reformulation of this:

$$\mathbb{A} = \prod_{\substack{v \text{ place} \\ \text{of } \mathbb{Q}}}' \mathbb{Q}_v = \mathbb{A}_f \times \underset{\mathbb{Q}_\infty}{\mathbb{R}} \qquad \text{where } \mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

$\rightsquigarrow$ $\boxed{L^2_G = L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/A_G)} \circlearrowleft G(\mathbb{A})$

A_G: if $S_G = Z(G)^{\text{split}}$, $A_G = S_G(\mathbb{R})^\circ$ conn component.

Also, $\Pi_G :=$ discrete automorphic rep'ns of $G(\mathbb{A})$.
$= \{$irrep π of $G(\mathbb{A})$ that are direct factors of $L^2_G\}$

$\rightsquigarrow$ $m(\pi)$ = multiplicity of π in irrep'n of L^2_G.
(choose a Haar measure on $G/Z(G)$).

Assume $G/Z(G)$ anisotropic. (*)

Then $G(\mathbb{Q})\backslash G(\mathbb{A})/A_G$ is compact.

$$\Rightarrow \quad \underset{\substack{\circlearrowleft \\ G(\mathbb{A})}}{L^2_G} = \widehat{\bigoplus_{\pi\in\Pi(G)}} \pi^{m(\pi)}.$$

· Adelic locally symmetric spaces

$M_K = G(\mathbb{Q})\backslash (X\times G(\mathbb{A}_f))/K$, K open compact subgrp of finite adelic points $G(\mathbb{A}_f)$.

Rmk If $G(\mathbb{A}_f) = \coprod_{i\in I} G(\mathbb{Q})x_i K$

then $\boxed{M_K = \coprod_i \Gamma_i\backslash X}$, $\Gamma_i = x_i K x_i^{-1}\cap G(\mathbb{Q})$.

this is a symmetric space.

But $\varprojlim_K M_K = G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f) \circlearrowleft G(\mathbb{A}_f)$ by right translation.

So $\varinjlim_K H^i(M_K) =: H^i \circlearrowleft G(\mathbb{A}_f)$.

Big Thm (Matsushima) Assuming (*) before.

As $G(\mathbb{A}_f)$-rep'ns, $\boxed{H^i \simeq \bigoplus_{\pi\in\Pi(G)} \pi_f\otimes H^i(\mathfrak{g}, K_\infty, \pi_\infty)^{m(\pi)}}$

$\pi = \pi_f\otimes\pi_\infty = \pi^\infty\otimes\pi_\infty$
with $G(\mathbb{A}) = G(\mathbb{A}_f)\times G(\mathbb{R})$

$\mathfrak{g} = \mathrm{Lie}\, G(\mathbb{R})$.

Or $H^i(M_K) \simeq \bigoplus_{\pi\in\Pi(G)} \pi_f^K \otimes H^i(\mathfrak{g}, K_\infty, \pi_\infty)^{m(\pi)}$

as $\mathcal{H}_K = C_c(K\backslash G(\mathbb{A}_f)/K, \mathbb{C})$-modules.

§2 Motivation from modular curves

$G = GL_2$, M_K = modular curves = moduli spaces of elliptic curves.

$\rightsquigarrow$ M_K's are varieties def'd $/\mathbb{Q}$.

Hence $\varinjlim_K H^i_{Betti}(M_K, \mathbb{C}) \simeq \varinjlim_K H^i_{et}(M_{K,\bar{\mathbb{Q}}}, \bar{\mathbb{Q}}_\ell)$

$\circlearrowleft$ $G(\mathbb{A}_f) \times Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

via Hecke correspondences

If π comes from a modular form,

(Deligne:) $\underline{H^1[\pi_f]} =: \boxed{\sigma(\pi)} \leftarrow Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ rep'n of dim 2.

the π_f-isotypic component

Can check at every place that $\sigma(\pi) = GL(\pi)$.

Questions (keynote)

(1) When is $\Gamma\backslash X$ (or M_K) an algebraic variety over $\mathbb{C}$?

(2) When it is, can we descend it to $\mathbb{Q}$ or other number field?

Answer to (1): $X = G(\mathbb{R})/K_\infty$

We say that X is a Hermitian symmetric domain (HSD) if X has a complex structure and a Hermitian metric s.t. $G(\mathbb{R})$ acts by holomorphic isometries.

e.g. Poincaré upper-half space $(d \geq 1)$

$$\mathfrak{h}_d^+ = \{X \in M_{d\times d}(\mathbb{C}) \mid {}^tX = X \ \& \ Im(X) > 0\}.$$

$$\circlearrowleft$$

$$Sp_{2d}(\mathbb{R}) := \{g \in GL_{2d}(\mathbb{C}) \mid {}^tg \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}\}$$ symplectic grp

$\rightsquigarrow$ Can check

$$\begin{array}{c c} & \begin{array}{cc} d & d \end{array} \\ \begin{array}{c} d \\ d \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \cdot X = (AX+B)\cdot(CX+D)^{-1}$$

(transitive action).

Take $K_\infty := Stab_{Sp_{2d}(\mathbb{R})}(iI_d) = O(2d) \cap Sp_{2d}(\mathbb{R})$.

So $\mathfrak{h}_d^+ = Sp_{2d}(\mathbb{R})/K_\infty$.

bounded, carries Bergman metric.

Bounded realization $\mathfrak{h}_d^+ \xrightarrow{\sim} \mathcal{D}_d := \{X \in M_d(\mathbb{C}) \mid {}^tX = X,\ I_d - X^*X > 0\}$

$$X \longmapsto (iI_d - X)(iI_d + X)^{-1}$$

Thm Suppose that $G(\mathbb{R})$ is connected adjoint.

going to be the group of diffeomorphisms on X.

Then X is a HSD iff $\exists\, u : U(1) \longrightarrow G(\mathbb{R})$ s.t.

(a) The characters of $U(1)$ in $\mathrm{Lie}\, G(\mathbb{C}) \circlearrowleft \mathrm{Ad} \circ u$.

(b) $\mathrm{Int}(u(i))$ is a Cartan involution of $G(\mathbb{R})$.

($\{g \in G(\mathbb{C}) \mid g = u(i)\, \bar{g}\, u(i)^{-1}$ is compact inner form$\}$).

Moreover, we may assume that $K_\infty = \mathrm{Cent}_{G(\mathbb{R})}(u)$

so $X = \{G(\mathbb{R})\text{-conj classes of } u\}$.

Example $G = PSp_{2d}$, $u : a + ib \longmapsto \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}$.

Rmk The $G(\mathbb{R})$ having such a u are classified:

(i) Type A: $PSU(p,q)$

(ii) no types G_2, F_4, E_8.

· Compatification.

Thm (Baily-Borel, Borel) If X is a HSD, then $\forall \Gamma$,

$\Gamma \backslash X$ has a unique structure of quasi-proj alg var $/\mathbb{C}$.

§3 Siegel modular varieties

(a higher-dim'l generalization for modular curves).

A a.v. $/\mathbb{C}$, $\dim A = d$ fixed.

$A \simeq \mathrm{Lie}(A)/\Lambda$, $\mathrm{Lie}\, A \simeq \mathbb{C}^d$, $\Lambda = \pi_1(A) \simeq H_1(A, \mathbb{Z})$.

$\rightsquigarrow A^\vee = \mathrm{Lie}(A^\vee)/\Lambda^\vee$. $\mathrm{Lie}(A^\vee) = \{$semilinear forms on $\mathrm{Lie}(A)\}$

& $\Lambda^\vee = \{ \ell \in \mathrm{Lie}(A^\vee) \mid \mathrm{Im}\, \ell(\Lambda) \subseteq \mathbb{Z} \}$.

$$\{\text{polarization } \lambda : A \to A^\vee\} \simeq \left\{ \begin{array}{c} \text{pos. def. Hermitian forms } H \text{ on } \mathrm{Lie}(A) \\ \text{s.t. } \mathrm{Im}(H(\Lambda \times \Lambda)) \subseteq \mathbb{Z} \end{array} \right\}$$

$$(\lambda_H : v \mapsto H(v, \cdot)) \longleftarrow\!\!\shortmid \; H$$

Weil pairing $f_{\lambda_H} : A[n] \times A[n] \xrightarrow{\mathrm{id} \times \lambda} A[n] \times A^\vee[n] \longrightarrow \mu_n(\mathbb{C})^t$

$$(v, w) \longmapsto \exp(-2\pi i n H(v, w)).$$

<u>Prop</u> $\mathfrak{h}_d^+ \simeq \{(A, \lambda, \eta_{\mathbb{Z}})\}/\sim$.

- A a.v. $/\mathbb{C}$ of dim d
- $\lambda : A \xrightarrow{\sim} A^\vee$ principal polarization
- $\eta_{\mathbb{Z}} : \underset{\Lambda}{\underset{\wr\wr}{H_1(A, \mathbb{Z})}} \xrightarrow{\sim} \mathbb{Z}^{2d}$ symplectic iso.

$$X \in \mathfrak{h}_d^+ \longmapsto (A_X, \lambda_X, \eta_{X,\mathbb{Z}})$$

where $A_X = \mathbb{C}^d / (\mathbb{Z}^d + X\mathbb{Z}^d)$

$\lambda_X = \lambda_{H_X}$, $H_X = (\mathrm{Im}\, X)^{-1}$,

$\eta_{X,\mathbb{Z}} : H_1(A, \mathbb{Z}) = \mathbb{Z}^d + X\mathbb{Z}^d \xrightarrow{\sim} \mathbb{Z}^{2d}$

$e_i \mapsto e_i$, $Xe_i \mapsto e_{i+d}$ $(1 \le i \le d)$.