Shimura varieties (3/3)

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§6 A big picture of Shimura varieties

<u>Recall</u> A Shimura datum (G,h) with $h: \mathbb{C}^\times \to G(\mathbb{R})$

$$\rightsquigarrow (M_K(G,h))(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

X: $G(\mathbb{R})$-conj classes of h.

Reflex field $E = E(G,h)$ / $\mathbb{Q}$ finite.

$E \hookrightarrow \mathbb{C}$

	Canonical model?	Moduli interpretation?	Integral model?
{General Shimura varieties} ∪	Yes! G of type A, B, C, D, E_6, E_7 (A: unitary.)	?	?
abelian type ∪	A, B, C, and some type D	moduli space of motives	Yes (Kisin).
Hodge type ∪	same as the abelian but with some tech conditions	Polarized AVs + Hodge tensors	Yes.
PEL type ∪	type A, C, D	polarized AVs + multiplication by some $\mathbb{Q}$-algebra	
{Siegel modular varieties}	$G = GSp_{2d}$	With moduli interpretation	Yes, $\mathbb{Z}[\frac{1}{n}]$ in level $K(n)$.

$K(n) = \mathrm{Ker}(GSp_{2d}(\hat{\mathbb{Z}}) \twoheadrightarrow GSp_{2d}(\mathbb{Z}/n\mathbb{Z}))$

<u>Caution</u> Abelian type ≠ moduli space of AVs. (it's Hodge type!)

- <u>Hodge type Shimura datum</u>

  (G,h) s.t. $\exists d$, $(G,h) \hookrightarrow (GSp_{2d}, h_d)$ (Siegel Shimura datum).

- <u>Abelian type Shimura datum</u>

  (G',h') s.t. $(G',h') \leftarrow (G,h) \hookrightarrow (GSp_{2d}, h_d)$

  isom on $(G'_{ad}, h'_{ad}) \simeq (G_{ad}, h_{ad})$.

- <u>PEL type</u>

E.g. $\mathbb{Q} \overset{?}{-} F_0 \overset{2}{-} F$, F_0 tot real, F CM. $(Gal(F/F_0) = \{id, z \mapsto \bar{z}\})$.

B/F central simple algebra

$*$ positive involution $(Tr_{B/\mathbb{R}}(x^* x) > 0, \forall x \neq 0)$ extending $z \mapsto \bar{z}$.

Take $G/\mathbb{Q}$ to be

$$G = \{g \in B^\times \mid gg^* = c(g)1, \exists c(g) \in GL_1\} \xrightarrow{c} GL_1.$$

$$\cup\!\!\!\mathsf{I}$$

$$G_0 = \ker c = Res_{F_0/\mathbb{Q}} \underbrace{\boxed{U_{F_0}(B,*)}}_{=:U}.$$

$\rightsquigarrow$ $G_0(\mathbb{R}) = \prod_{\tau: F_0 \hookrightarrow \mathbb{R}} U(p_\tau, q_\tau)$, $p_\tau + q_\tau = n = \sqrt{\dim_F B}$.

Take $h: \mathbb{C}^\times \longrightarrow G(\mathbb{R})$ to be

$$h(z) = \left(\begin{pmatrix} z I_{p_\tau} & 0 \\ 0 & \bar{z} I_{q_\tau} \end{pmatrix} \right)_{\tau: F_0 \hookrightarrow \mathbb{R}}.$$

If n is odd, we can choose U_{F_v} freely, $\forall v$ place of F.

If n is even, there's a parity condition.

<u>Rmk</u> If B is a division alg. $M_K(G,h)$ are compact

(Kottwitz simple Shimura varieties).

<u>§7 Integral models</u>

$M_K(G,h)$ on $E = E(G,h)$.

We expect a nice integral model $\mathscr{S}_K$ over $\mathcal{O}_{E,\wp}$ when $\wp | p$ prime ideal (basically everything is unramified at p).

s.t. $K = K_p K^p$, where

$K_p \leq G(\mathbb{Q}_p)$ hyperspecial ($\Rightarrow G_{\mathbb{Q}_p}$ unramified).

$K^p \leq G(\mathbb{A}_f^p)$

- <u>Characterization</u> (Milne / Moonen).

$\forall\, S \to \operatorname{Spec} \mathcal{O}_{E,p}$ a nice test scheme

any $S_E \longrightarrow M_K(G,h)$ extends to $\mathcal{S} \to \mathscr{S}_K$, $\mathscr{S}_K$ smooth.

<u>By Kisin</u>: (i) Gave a class of test schemes

(ii) Constructed integral models in abelian type.

Idea of construction:

if (G,h) is of Hodge type,

$$
\begin{tikzcd}
M_K(G,h) \arrow[r, hook, "\text{closed imm}", "\text{(Deligne)}"'] \arrow[d, "\text{(no normalization)}"'] & M_{K'}(GSp_{2d}, h_d) \arrow[d] \\
\mathscr{S}_K = \overline{M_K(G,h)} \arrow[r, hook, "\mathscr{S}_{K', \mathcal{O}_{E,p}}"] & \mathscr{S}_{K', \mathcal{O}_{E,p}}
\end{tikzcd}
$$

- <u>Kottwitz conjecture</u>

(G,h) with G^{der} anisotropic. (full generality ver.)

$\boxed{H^i} := \varinjlim_K H^i_{et}(M_K(G,h)_{\bar{E}}, \overline{\mathbb{Q}}_\ell)$

$\circlearrowleft$

$G(\mathbb{A}_f) \times Gal(\bar{E}/E)$

π_f irred. rep'n of $G(\mathbb{A}_f)$

$\rightsquigarrow H^i[\pi_f] = \pi_f$-isotypic component

$\circlearrowleft$

$Gal(\bar{E}/E)$ with $\dim = \sum_{\pi_\infty} m(\pi) \dim H^i(\mathfrak{g}, K_\infty; \pi_\infty)$.

s.t. $\pi = \pi_f \otimes \pi_\infty \in \Pi(G)$

Conj (Kottwitz) Assume $d = \dim M_K(G,h)$

$$\sum_{i\geq 0} (-1)^i [H^i[\pi_f]](d/2) = \sum_{\substack{\pi_\infty \text{ s.t.} \\ \pi = \pi_f \otimes \pi_\infty \in \Pi(G) \\ a(\pi) \in \mathbb{Z}}} a(\pi) \cdot [r_\mu \circ \varphi_\pi |_{Gal(\bar{E}/E)}] .$$

① φ_π = conjectural Langlands parameter of π.

$$\varphi_\pi : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow {}^L G(\bar{\mathbb{Q}}_\ell)$$

Actually RHS only depends on π_f.

② What is r_μ?

$$h : \mathbb{C}^\times \longrightarrow G(\mathbb{R})$$

$\rightsquigarrow$ $\mu_h : \mathbb{C}^\times \longrightarrow G(\mathbb{C})$ conj class defined $/E$

$\rightsquigarrow$ $r_\mu : \hat{G} \longrightarrow GL(V_\mu)$ with highest weight "μ_h".

r_μ extends to $\hat{G} \rtimes Gal(\bar{E}/E)$.

E.g. $G = GSp_{2d}$,

$$h : a+ib \longmapsto \begin{pmatrix} aI_d & -bI_d \\ bI_d & aI_d \end{pmatrix}.$$

$\hat{G} = GSpin_{2d+1}(\bar{\mathbb{Q}}_\ell)$, r_μ = spin rep'n of $\hat{G}$.

In general, $\hat{G} = GL_1(\bar{\mathbb{Q}}_\ell) \times \prod_{\tau : F_0 \hookrightarrow \mathbb{R}} GL_n(\bar{\mathbb{Q}}_\ell)$

$$r_\mu = \chi \otimes \bigotimes_\tau r_{\mu,\tau} , \quad r_{\mu,\tau} = \Lambda^{p_\tau} \text{ (standard)}.$$

§8 Applications

* L-function of $M_K(G,h)$
  ($\rightsquigarrow$ meromorphic continuation, functional equation)
* Constructing φ_π for some π (global Langlands)

(a long story by Clozel + Kottwitz ...)

<u>How to prove?</u>

(1) Specialization via integral mode.

$$H^i(M_K(G,h)_{\bar{E}}) \simeq H^i(\mathcal{M}_K, \bar{\mathcal{O}}_{E/p})$$

$$\circlearrowleft$$

$$\mathrm{Gal}(\bar{E}_p/E_p) \times \underbrace{G(\mathbb{A}_F^p)}$$

Hecke corr outside of p.

(2) Lefschetz fixed point formula

+ counting points (Langlands-Rapoport conj.)

Abelian type: ok. by Kisin

PEL type: à la Kottwitz.

(3) trace formula

(2) $\rightsquigarrow$ (3): stabilization (some sort of the fundamental lemma).