What does geometric Langlands mean to a number theorist? (1/2)

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(Joint with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, Varshavsky.)

<u>Goal</u> Give an arithmetic version of the geom Langlands conj's.

<u>Geometric setting</u>

k alg closed field, X/k sm proj connected curve.

G/k a split red gp.

$\rightsquigarrow$ Bun_G moduli stack of G-bundles on X.

<u>Auxiliary notation</u> $e = \bar{\mathbb{Q}}_\ell$. $\check{G}/e$.

$S/k \rightsquigarrow Shv(S)^c$ = DG category of constructible e-sheaves

Have $Shv(S) = Ind(Shv(S)^c)$.

Extend to stacks in usual way.

$Lisse(S) \subseteq Shv(S)$

subcat where all (perverse) cohomologies of $\mathcal{F} \in Shv(S)$ can be written as a colim of (finite rank) (usual) lisse sheaves.

(In their papers, called $Qlisse(S)$.)

<u>Arithmetic setting</u>

$k = \bar{\mathbb{F}}_q$, $X/\mathbb{F}_q$.

$\rightsquigarrow Frob_X : X \longrightarrow X$ (over k) $\quad$ $\mathbb{Q}$-Frobenius.

$X(\mathbb{F}_q) = X^{Frob_X = id}$.

Similarly, $Frob_{Bun_G} : Bun_G \longrightarrow Bun_G$.

We define:

(Geom setting) $LS^{restr}_{\check{G}} \longrightarrow \mathrm{Spec}\, e$

S/e test scheme, a map

$S \longrightarrow LS^{restr}_{\check{G}} \overset{\text{def'n}}{\Longleftrightarrow}$ a (right t-exact) symmetric monoidal functor $\mathrm{Rep}\,\check{G} \longrightarrow \mathrm{Lisse}(X) \otimes \mathrm{QCoh}(S)$

$\qquad = A\text{-mod}(\mathrm{Lisse}(X))$ if $S = \mathrm{Spec}\, A$

<u>More arithmetic setting</u>:

Obtain a "Frobenius"

$$\begin{tikzcd} LS^{restr}_{\check{G}} \arrow[rr, "\Phi"] \arrow[dr] & & LS^{restr}_{\check{G}} \arrow[dl] \\ & \mathrm{Spec}\, \overline{\mathbb{Q}}_\ell & \end{tikzcd}$$

$\Phi(\sigma) = \mathrm{Frob}^*_X(\sigma)$.

Set $LS^{arithm}_{\check{G}} = (LS^{restr}_{\check{G}})^{\Phi = id}$.

Explicitly, this has a functor-of-points description lisse before, but we use Weil local systems on X instead:

$$\mathrm{WeilLisse}(X) := \mathrm{Lisse}(X)^{\mathrm{Frob}^* = id}.$$

<u>Representability thms</u>

- $LS^{restr}_{\check{G}}$ is a formal alg stack w/ infinitely many connected comps.
- $LS^{arithm}_{\check{G}}$ is a quasi-compact alg stack.

<u>Goal</u> Construct a certain quasi-coherent sheaf

$$\mathrm{Drinf} \in \mathrm{QCoh}(LS^{arithm}_{\check{G}}) \quad \text{(Drinfeld alg stack)}$$

with $\Gamma(\mathrm{Drinf}) = \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbb{F}_q))$.

<u>Thesis</u> Drinf is a better object to study than the RHS.

sth like "space of autom forms".

Goal 2 Drinf can be computed using geom Langlands conj (GLC).

Traces $\mathcal{Y}$ finite set. $F : \mathcal{Y} \to \mathcal{Y}$

$\# \mathcal{Y}^F = tr(F^* : \mathrm{Fun}\, \mathcal{Y} \to \mathrm{Fun}\, \mathcal{Y})$.

Geometrically $\mathcal{Y}$ stack, $F : \mathcal{Y} \to \mathcal{Y}$

$$\mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{\Delta^*} \mathrm{Shv}(\mathcal{Y}) \xrightarrow{C_c} \mathrm{Set}$$

tr^{geom}

$\rightsquigarrow (F \times id)^*(\Delta_! e_{\mathcal{Y}}) = \mathrm{Graph}_{F,!}(e) \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$

its geom trace is $C_c(\mathcal{Y}^{F=id})$.

For $F = \mathrm{Frob}_{\mathcal{Y}}$, $\mathcal{Y}^{F=id} = \mathcal{Y}(\mathbb{F}_q)$.

$$C_c(\mathcal{Y}(\mathbb{F}_q)) = \mathrm{Fun}_c(\mathcal{Y}(\mathbb{F}_q)).$$

In case $\mathcal{Y} = \mathrm{Bun}_G$, there's many operators acting on $\mathrm{Shv}(\mathrm{Bun}_G)$: $\boxed{\text{Hecke operators}}$

Background on Hecke functors

Step 0 $V \in \mathrm{Rep}\, \check{G} \rightsquigarrow H_V : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X)$

with the property that its x-fiber at a pt $x \in X$ is "Hecke functor at $x \in X$".

Given by an object $\mathcal{K}_V \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G \times X)$

an abstract str of "kernel".

Step 1 $V \in \mathrm{Rep}\, \check{G}^I \rightsquigarrow H_V : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_G \times X^I)$.

and $\mathcal{K}_V \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G \times X^I)$.

Step 2 (Variant) $V \in \mathrm{Rep}\, \check{G}^I$, $\mathcal{F} \in \mathrm{Shv}(X^I)$

$\rightsquigarrow H_{V,\mathcal{F}} : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$

by taking H_V and then $\overset{*}{\otimes}$ with $\mathcal{F}$ along the second factor and pushing forward to Bun_G.

<u>Rem</u> up to shift, can do $\overset{!}{\otimes}$ & $*$-pushforward.

Again: Have $K_{V,\mathcal{F}} \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$.

$K \in \mathrm{Shv}(\mathcal{Y} \times \mathcal{Z}) \rightsquigarrow F_K : \mathrm{Shv}(\mathcal{Y}) \longrightarrow \mathrm{Shv}(\mathcal{Z})$

& $F_K(\mathcal{F}) := p_{2,!}(p_1^*(\mathcal{F}) \overset{*}{\otimes} K)$

$$\begin{tikzcd} & \mathcal{Y} \times \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{Y} & & \mathcal{Z} \end{tikzcd}$$