What does geometric Langlands mean to a number theorist? (2/2)

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(Continue on ...)

Universal source:

$$\operatorname{Rep} \check{G}_{Ran} := \underset{\mathrm{TwArr}(fSets)}{\operatorname{colim}} \; \mathrm{Shv}(X^I) \otimes \operatorname{Rep} \check{G}^J$$

← DGCat_{cont}

twisted arrows of finite sets.

What this means:

- $\forall I$, have $\mathrm{ins}_I : \mathrm{Shv}(X^I) \otimes \operatorname{Rep} \check{G}^I \longrightarrow \operatorname{Rep} \check{G}_{Ran}$
- basic relations: $I \longrightarrow J$

⇝ a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Shv}(X^I) \otimes \operatorname{Rep} \check{G}^J \arrow[dr, "\mathrm{ins}_J"] & \\
\mathrm{Shv}(X^I) \otimes \operatorname{Rep} \check{G}^I \arrow[ur, "\mathrm{id} \otimes \mathrm{multi}"] \arrow[dr, "\Delta_* \otimes \mathrm{id}"'] \arrow[rr, phantom, "\circlearrowright"] & & \operatorname{Rep} \check{G}_{Ran}. \\
 & \mathrm{Shv}(X^I) \otimes \operatorname{Rep} \check{G}^I \arrow[ur, "\mathrm{ins}_I"'] &
\end{tikzcd}$$

+ higher compatibilities.

$\{1,2\} \longrightarrow *$ induces

$$\begin{tikzcd}
 & \mathrm{Shv}(X) \otimes \operatorname{Rep} \check{G} \arrow[dr] & \\
\mathrm{Shv}(X) \otimes \operatorname{Rep} \check{G}^2 \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circlearrowright"] & & \operatorname{Rep} \check{G}_{Ran} \\
 & \mathrm{Shv}(X^2) \otimes \operatorname{Rep} \check{G}^2 \arrow[ur] &
\end{tikzcd}$$

Form of Hecke action we want:

$$\mathrm{Rep}\, \check{G}_{Ran} \curvearrowright \mathrm{Shv}(Bun_G)$$

really acts by kernels

$$\mathrm{Rep}\, \check{G}_{Ran} \longrightarrow \mathrm{Shv}(Bun_G \times Bun_G) \qquad \text{(monoidal)}$$
$$V \longmapsto K_V.$$

E.g. $V = \mathrm{triv} \Rightarrow K_V = \Delta_! \underline{e}_{Bun_G}$.

<u>Defn</u> The functor $\mathrm{Sht} : \mathrm{Rep}\, \check{G}_{Ran} \longrightarrow \mathrm{Vect}$ is the composition

$$\mathrm{Rep}\, \check{G}_{Ran} \longrightarrow \mathrm{Shv}(Bun_G \times Bun_G) \xrightarrow{(id \times Frob)^*} \mathrm{Shv}(Bun_G \times Bun_G) \xrightarrow{tr^{geom}} \mathrm{Vect}$$

This Sht is rigged so $\mathrm{Sht}(\mathrm{triv}) = \mathrm{Func}(Bun_G(\mathbb{F}_q))$.

<u>Relationship to usual shtukas</u>

(1) Fix a finite set I. Define

$$\mathrm{Sht}^{true} : \mathrm{Rep}\, \check{G}^I \longrightarrow \mathrm{Shv}(X^I)$$
$$V \longmapsto \underbrace{(\text{"}tr^{geom} \otimes id\text{"})}_{p_{2,!}(\Delta_{Bun_G} \times id_{X^I})^*} (id \times Frob \times id)^*(K_V)$$

i.e. $\underset{\substack{\rotatebox{90}{$\in$} \\ K_V}}{\mathrm{Shv}(Bun_G \times Bun_G \times X^I)} \xrightarrow{(\mathrm{Graph}\, Frob_{Bun_G} \times id_{X^I})^*} \mathrm{Shv}(Bun_G \times X^I) \xrightarrow{p_{2,!}} \mathrm{Shv}\, X^I$

(2) Given a functor $F : \mathrm{Rep}\, \check{G}_{Ran} \longrightarrow \mathrm{Vect}$, we obtain functors $\forall I$:

$$F_I : \mathrm{Rep}\, \check{G}_I \longrightarrow \mathrm{Shv}(X^I)$$

characterized by $\mathcal{F} \in \mathrm{Shv}(X^I)$, $V \in \mathrm{Rep}\, \check{G}^I$. Cohomology $C^\cdot$

$$F(\mathrm{ins}_I(\mathcal{F} \boxtimes V)) = C^\cdot(X^I, \mathcal{F} \overset{!}{\otimes} F_I(V)).$$

<u>Exercise</u>: $\forall \mathcal{F} \in \mathrm{Shv}(X^I)$, $V \in \mathrm{Rep}\, \check{G}^I$,

$$C^\cdot(X^I, \mathrm{Sht}^{true}(V) \overset{*}{\otimes} \mathcal{F}) = C^\cdot(X^I, \mathrm{Sht}_I(V) \overset{!}{\otimes} \mathcal{F}) = \mathrm{Sht}(\mathrm{ins}_I(\mathcal{F} \boxtimes V)).$$

Thm (Xue) $\mathrm{Sht}_I^{\mathrm{true}}(V) \in \mathrm{Lisse}(X^I)$.

It follows that

$$\mathrm{Sht}_I^{\mathrm{true}}(V) \overset{*}{\otimes} \mathcal{F} = \mathrm{Sht}_I^{\mathrm{true}}(V) \overset{!}{\otimes} \mathcal{F}\,[2|I|]$$

$$\Rightarrow \quad \mathrm{Sht}_I^{\mathrm{true}}(V) = \mathrm{Sht}_I(V)\,[-2|I|].$$

Relation to $\mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ & $\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}$:

Tool $\exists$ canonical functor

$$\mathrm{Loc} : \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}).$$

Idea $x_1, \dots, x_n \in X$, $V_1, \dots, V_n \in \mathrm{Rep}\,\check{G}$, $\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$,

$\rightsquigarrow$ vector space $\bigotimes_{i=1}^{n} (V_i, \sigma)_{x_i}$

Properly, $\quad \mathrm{Can} : \mathrm{Rep}\,\check{G} \longrightarrow \mathrm{Lisse}(X) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$.

$$\rightsquigarrow \quad \mathrm{Can}^I : \mathrm{Rep}\,\check{G}^I \longrightarrow \mathrm{Lisse}(X)^{\otimes I} \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})^{\otimes I}$$

$$\downarrow$$

$$\mathrm{Lisse}(X^I) \otimes \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}).$$

(Like before) $\rightsquigarrow$ $\mathrm{Rep}\,\check{G}^I \otimes \mathrm{Shv}(X^I) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$

by def'n: $\mathrm{Loc} \circ \mathrm{ins}_I$.

Thm ([AGKRRV3, Cor 2.3.4])

$F : \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \longrightarrow \mathrm{Vect}$ factors

and it factors uniquely as

$$\begin{tikzcd} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \arrow[rr, "F"] \arrow[d, "\mathrm{Loc}"'] & & \mathrm{Vect} \\ \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}) \arrow[urr, dashed, "F^{\mathrm{Coh}}"'] & & \end{tikzcd}$$

if and only if $\forall I$, the functor $F_I : \mathrm{Rep}\,\check{G}^I \longrightarrow \mathrm{Shv}(X^I)$

takes values in $\mathrm{Lisse}(X^I)$.

Cor $\exists$ a functor $\mathrm{Sht}^{\mathrm{enh}} : \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \longrightarrow \mathrm{Vect}$.

Generalities $\Rightarrow$ $\mathrm{Sht}^{\mathrm{enh}}(\mathcal{F}) = \Gamma_!(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}, \mathcal{F} \otimes \mathrm{Drinf}^{\mathrm{geom}})$

for some $\mathrm{Drinf}^{\mathrm{geom}} \in \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

Claim Geom Langlands predicts that

$$\mathrm{Drinf} = \omega_{\mathrm{LS}^{\mathrm{restr}}_{\check{G}}}.$$

Picture $\sigma \in \mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ is elliptic / discrete.

$\mathbb{B}\mathrm{Aut}(\sigma)$ is a connected cpt of $\mathrm{LS}^{\mathrm{arithm}}_{\check{G}}$.

$\rightsquigarrow$ 1-dim'l summand of $\mathrm{Func}(\mathrm{Bun}_G(\mathbb{F}_q))$.

Messier near other parameters:

Let's be in the geom setting.

There's a certain subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subseteq \mathrm{Shv}(\mathrm{Bun}_G)$$

Conj $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{Spec}}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$.

More about the structure of the LHS:

- $\exists\, \mathcal{R} \in \mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ canonical object.

  $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ turns out to be canonically self-dual as a DG category.

s.t.
$$\begin{array}{ccc} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \otimes \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} & \ni & \mathrm{unit} \\ \big\downarrow \mathrm{id} \otimes \mathrm{Loc} & & \\ \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \xrightarrow{\mathrm{id} \otimes \Gamma_!} \mathrm{Rep}\,\check{G}_{\mathrm{Ran}} & \ni & \mathcal{R}. \end{array}$$

Calculate: $\mathrm{Loc}(\mathcal{R}) = \mathcal{O}_{\mathrm{LS}_{\check{G}}^{\mathrm{restr}}}$.

We show $\mathcal{R}_*(-) : \mathrm{Shv}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$

maps into $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

and gives the right adjoint to the embedding.

<u>Application</u> $K \in \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$

$$\begin{tikzcd}
\rightsquigarrow \ \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \arrow[r, hook] \arrow[rrd, dashed, "F_{K,\mathrm{Nilp}}"'] & \mathrm{Shv}(\mathrm{Bun}_G) \arrow[r, "F_K"] & \mathrm{Shv}(\mathrm{Bun}_G) \arrow[d, "\mathcal{R}_*(-)"] \\
& & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).
\end{tikzcd}$$

<u>Thm</u> ([AGKRRV2]) $\mathrm{tr}(F_{K,\mathrm{Nilp}}) = \mathrm{tr}^{\mathrm{geom}}(\mathcal{R} \circ K \circ \mathcal{R}) = \mathrm{tr}^{\mathrm{geom}}(\mathcal{R} \circ K)$.

<u>Cor</u> $V \in \mathrm{Rep}\, \check{G}_{\mathrm{Ran}}$, then

thm

$$\mathrm{tr}\big((V_*(-)) \circ \mathrm{Frob}^*_{\mathrm{Bun}_G} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\big)$$

$$= \mathrm{tr}^{\mathrm{geom}}(\mathcal{R} \circ K_V \circ (\mathrm{id} \times \mathrm{Frob})^*)$$

thm

$$= \mathrm{tr}^{\mathrm{geom}}(K_{\mathcal{R} * V} \circ (\mathrm{id} \times \mathrm{Frob})^*)$$

$$= \mathrm{Sht}(\mathcal{R} * V)$$

V

$$= \mathrm{Sht}^{\mathrm{enh}}(\boxed{\mathrm{Loc}(\mathcal{R})} \otimes \mathrm{Loc}(V)).$$

$$= \mathrm{Sht}^{\mathrm{enh}}(\mathrm{Loc}(V)) = \mathrm{Sht}(V).$$

<u>See</u> categorical traces of Frob-Hecke ops

$\rightsquigarrow$ shtuka cohomology on $\mathrm{Shv}_{\mathrm{Nilp}}$.

When $V = \mathrm{triv}$, have

$$\mathrm{tr}(\mathrm{Frob}^*_{\mathrm{Bun}_G} \circlearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) = \mathrm{Func}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

Can show: $\mathrm{tr}(\Phi^* \curvearrowright \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}))$

$= \mathrm{tr}(\Phi^* \curvearrowright \mathrm{IndCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}}))$

$= \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arithm}}, \omega)$.