

What does geometric Langlands mean to a number theorist? (2/2)

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(Continue on...)

Universal source:

$$\text{Rep } \check{G}_{\text{Ran}} := \underset{\text{TwArr}(\text{fSets})}{\text{colim}} \text{Sh}(X^I) \otimes \text{Rep } \check{G}^I$$

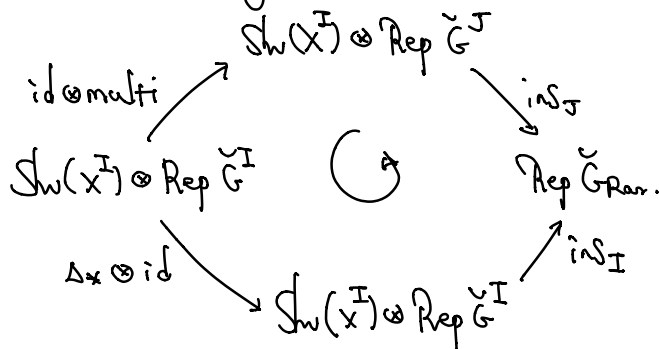
← DG Cat_{cont}

twisted arrows of finite sets.

What this means:

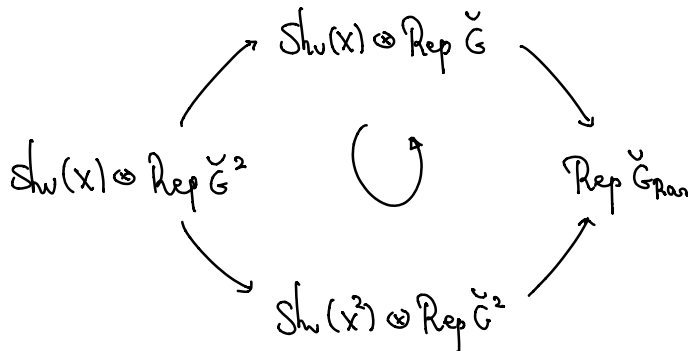
- $\forall I$, have $\text{ins}_I: \text{Sh}(X^I) \otimes \text{Rep } \check{G}^I \rightarrow \text{Rep } \check{G}_{\text{Ran}}$
- basic relations: $I \rightarrow J$

↪ a commutative diagram



+ higher compatibilities.

$\{1, 2\} \rightarrow *$ induces



Form of Hecke action we want:

$$\begin{array}{ccc} \text{Rep } \check{G}_{\text{Ran}} & \subset & \text{Shw}(\text{Bun}_G) \\ \text{really acts by kernels} & & \\ \text{Rep } \check{G}_{\text{Ran}} & \longrightarrow & \text{Shw}(\text{Bun}_G \times \text{Bun}_G) \quad (\text{monoidal}) \\ \mathcal{V} & \longmapsto & \text{Ker}. \end{array}$$

E.g. $\mathcal{V} = \text{triv} \Rightarrow \text{Ker} = \Delta! \mathbb{L}_{\text{Bun}_G}$.

Defn The functor $\text{Sht}: \text{Rep } \check{G}_{\text{Ran}} \rightarrow \text{Vect}$ is the composition

$$\text{Rep } \check{G}_{\text{Ran}} \rightarrow \text{Shw}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{(\text{id} \times \text{Frob})^*} \text{Shw}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{\text{tr}^{\text{geom}}} \text{Vect}$$

This Sht is rigged so $\text{Shw}(\text{triv}) = \text{Func}(\text{Bun}_G(\mathbb{F}_q))$.

Relationship to usual shukas

(1) Fix a finite set I . Define

$$\begin{array}{ccc} \text{Sht}^{\text{true}}: \text{Rep } \check{G}^I & \longrightarrow & \text{Shw}(X^I) \\ \mathcal{V} & \longmapsto & (\text{tr}^{\text{geom}} \otimes \text{id}) (\text{id} \times \text{Frob} \times \text{id})^* (\text{Ker}) \end{array}$$

$$\text{i.e. } \text{Shw}(\text{Bun}_G^m \times \text{Bun}_G \times X^I) \xrightarrow{\text{Graph}(\text{Frob}_{\text{Bun}_G} \times \text{id}_{X^I})^*} \text{Shw}(\text{Bun}_G \times X^I) \xrightarrow{\text{P}_2!} \text{Shw } X^I$$

$\text{P}_2! (\Delta_{\text{Bun}_G} \times \text{id}_{X^I})^*$
 $\text{P}_2! (\Delta_{\text{Bun}_G} \times \text{id}_{X^I})^*$

(2) Given a functor $F: \text{Rep } \check{G}_{\text{Ran}} \rightarrow \text{Vect}$, we obtain functors $\forall I$:

$$F_I: \text{Rep } \check{G}^I \rightarrow \text{Shw}(X^I)$$

characterized by $F \in \text{Shw}(X^I)$, $V \in \text{Rep } \check{G}^I$. cohomology C^*

$$F(\text{ins}_I(F \boxtimes V)) = C^*(X^I, F \otimes^! F_I(V)).$$

Exercise: $\forall F \in \text{Shw}(X^I)$, $V \in \text{Rep } \check{G}^I$,

$$C^*(X^I, \text{Sht}^{\text{true}}(V) \otimes^! F) = C^*(X^I, \text{Sht}_I(V) \otimes^! F) = \text{Sht}(\text{ins}_I(F \boxtimes V)).$$

Thm (Xue) $\text{Sh}_{\mathbb{Z}}^{\text{true}}(v) \in \text{Lisse}(X^{\mathbb{I}})$.

It follows that

$$\begin{aligned} \text{Sh}_{\mathbb{Z}}^{\text{true}}(v) \otimes^* F &= \text{Sh}_{\mathbb{Z}}^{\text{true}}(v) \otimes^! F [2|\mathbb{I}|] \\ \Rightarrow \text{Sh}_{\mathbb{Z}}^{\text{true}}(v) &= \text{Sh}_{\mathbb{Z}}^{\text{true}}(v) [-2|\mathbb{I}|]. \end{aligned}$$

Relation to $\text{LS}_{\mathbb{Z}}^{\text{restr}}$ & $\text{LS}_{\mathbb{Z}}^{\text{arith}}$:

Tool \exists canonical functor

$$\text{Loc}: \text{Rep } \check{G}_{\text{ran}} \longrightarrow \text{QCoh}(\text{LS}_{\mathbb{Z}}^{\text{restr}}).$$

Idea $x_1, \dots, x_n \in X$, $V_1, \dots, V_n \in \text{Rep } \check{G}$, $\sigma \in \text{LS}_{\mathbb{Z}}^{\text{restr}}$,

\hookrightarrow vector space $\bigotimes_{i=1}^n (V_i, \sigma)_{x_i}$

Property, $\text{Can}: \text{Rep } \check{G} \longrightarrow \text{Lisse}(X) \otimes \text{QCoh}(\text{LS}_{\mathbb{Z}}^{\text{restr}})$.

$$\hookrightarrow \text{Can}^{\mathbb{I}}: \text{Rep } \check{G}^{\mathbb{I}} \longrightarrow \text{Lisse}(X)^{\otimes \mathbb{I}} \otimes \text{QCoh}(\text{LS}_{\mathbb{Z}}^{\text{restr}})^{\otimes \mathbb{I}}$$

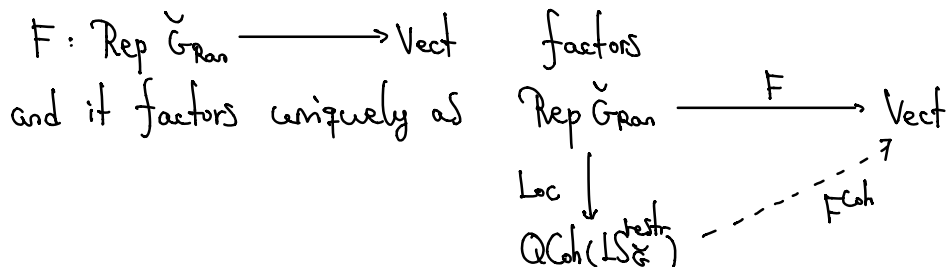
$$\downarrow$$

$$\text{Lisse}(X^{\mathbb{I}}) \otimes \text{QCoh}(\text{LS}_{\mathbb{Z}}^{\text{restr}}).$$

(Like before) $\hookrightarrow \text{Rep } \check{G}^{\mathbb{I}} \otimes \text{Sh}(X^{\mathbb{I}}) \longrightarrow \text{QCoh}(\text{LS}_{\mathbb{Z}}^{\text{restr}})$

by def'n: $\text{Loc} \circ \text{in}_{\mathbb{I}}$.

Thm (IAGKRRV3, Cor 2.3.4)



if and only if $\forall \mathbb{I}$, the functor $F_{\mathbb{I}}: \text{Rep } \check{G}^{\mathbb{I}} \longrightarrow \text{Sh}(X^{\mathbb{I}})$ takes values in $\text{Lisse}(X^{\mathbb{I}})$.

Cor \exists a functor $\text{Sh}^{\text{enh}} : \text{QCoh}(LS_{\mathbb{G}}^{\text{restr}}) \rightarrow \text{Vect}$.

Generalities $\Rightarrow \text{Sh}^{\text{enh}}(F) = \Gamma!(LS_{\mathbb{G}}^{\text{restr}}, F \otimes \text{Drf}^{\text{geom}})$
 for some $\text{Drf}^{\text{geom}} \in \text{QCoh}(LS_{\mathbb{G}}^{\text{restr}})$.

Claim Geom Langlands predicts that

$$\text{Drf}^{\text{geom}} = \omega_{LS_{\mathbb{G}}^{\text{restr}}}.$$

Picture $\sigma \in LS_{\mathbb{G}}^{\text{restr}}$ is elliptic / discrete.

$\text{BAut}(\sigma)$ is a connected cpt of $LS_{\mathbb{G}}^{\text{arithm}}$.

\hookrightarrow 1-diml summand of $\text{Func}(\text{Bun}_{\mathbb{G}}(\mathbb{F}_q))$.

Messier near other parameters:

Let's be in the geom setting.

There's a certain subcategory

$$\text{Sh}_{\text{nilp}}(\text{Bun}_{\mathbb{G}}) \subseteq \text{Sh}(\text{Bun}_{\mathbb{G}})$$

Conj $\text{Sh}_{\text{nilp}}(\text{Bun}_{\mathbb{G}}) \simeq \text{IndCoh}_{\text{Nilspec}}(LS_{\mathbb{G}}^{\text{restr}})$.

More about the structure of the LHS:

- $\exists \mathcal{R} \in \text{Rep } \check{\mathbb{G}}_{\text{ran}}$ canonical object.

$\text{Rep } \check{\mathbb{G}}_{\text{ran}}$ turns out to be canonically self-dual
 as a DG category.

$$\begin{array}{ccc} \text{s.t. } \text{Rep } \check{\mathbb{G}}_{\text{ran}} \otimes \text{Rep } \check{\mathbb{G}}_{\text{ran}} & \xrightarrow{\cong} & \text{unit} \\ \downarrow \text{id} \otimes \text{Loc} & & \searrow \\ \text{Rep } \check{\mathbb{G}}_{\text{ran}} \otimes \text{QCoh}(LS_{\mathbb{G}}^{\text{restr}}) & \xrightarrow{\text{id} \otimes \Gamma!} & \text{Rep } \check{\mathbb{G}}_{\text{ran}} \xrightarrow{\cong} \mathcal{R}. \end{array}$$

Calculate: $\text{Loc}(\mathcal{R}) = \mathbb{O}_{\mathbb{A}^1, \mathbb{F}_q}$.

We show $R_{\times}(-): \text{Shv}(\text{Bun}_G) \rightarrow \text{Shv}(\text{Bun}_G)$

maps into $\text{Shv}_{\text{nilp}}(\text{Bun}_G)$

and gives the right adjoint to the embedding.

Application $K \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)$

$$\begin{array}{ccc} \hookrightarrow \text{Shv}_{\text{nilp}}(\text{Bun}_G) & \hookrightarrow \text{Shv}(\text{Bun}_G) & \xrightarrow{F_K} \text{Shv}(\text{Bun}_G) \\ & \searrow \text{F}_{K, \text{nilp}} \text{ (dashed)} & \downarrow R_{\times}(-) \\ & & \text{Shv}_{\text{nilp}}(\text{Bun}_G). \end{array}$$

Thm ([AGRRV2]) $\text{tr}(\text{F}_{K, \text{nilp}}) = \text{tr}^{\text{geom}}(\mathcal{R} \circ K \circ \mathcal{R}) = \text{tr}^{\text{geom}}(\mathcal{R} \circ K)$.

Let $\mathcal{V} \in \text{Rep } \check{G}_{\text{ren}}$, then

$$\begin{aligned} \text{thm } \hookrightarrow & \text{tr}((\mathcal{V}_{\times}(-)) \circ \text{Frob}_{\text{Bun}_G}^* : \text{Shv}_{\text{nilp}}(\text{Bun}_G) \rightarrow \text{Shv}_{\text{nilp}}(\text{Bun}_G)) \\ & = \text{tr}^{\text{geom}}(\mathcal{R} \circ K_{\mathcal{V}} \circ (\text{id} \times \text{Frob})^*) \\ & = \text{tr}^{\text{geom}}(K_{\mathcal{R} \times \mathcal{V}} \circ (\text{id} \times \text{Frob})^*) \\ & = \text{Sht}(\mathcal{R} \times \mathcal{V}) \\ & = \text{Sht}^{\text{enh}}(\underline{\text{Loc}}(\mathcal{R}) \otimes \text{Loc}(\mathcal{V})). \\ & = \text{Sht}^{\text{enh}}(\text{Loc}(\mathcal{V})) = \text{Sht}(\mathcal{V}). \end{aligned}$$

See categorical traces of Frobenius-Hecke ops \hookrightarrow Shtuka cohomology on Shv_{nilp} .

When $\mathcal{V} = \text{triv}$, have

$$\text{tr}(\text{Frob}_{\text{Bun}_G}^* \hookrightarrow \text{Shv}_{\text{nilp}}(\text{Bun}_G)) = \text{Func}(\text{Bun}_G(\mathbb{F}_q)).$$

$$\begin{aligned}
\text{Can show: } & \Gamma(\mathbb{P}^k \curvearrowright \text{IndCoh}_{\text{NilpSpec}}(LS_G^{\text{restr}})) \\
&= \Gamma(\mathbb{P}^k \curvearrowright \text{IndCoh}(LS_G^{\text{restr}})) \\
&= \Gamma(LS_G^{\text{arithm}}, \omega).
\end{aligned}$$