The Langlands program and the moduli of bundles on the curve (1/3)

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§1 Local Langlands as geom Langlands on the FF curve

<u>Weinstein talks</u> · p-adic shtukas can be reinterpreted as modifications of G-bundles on the curve.

· moduli space of p-adic shtukas should realize the local Langlands correspondence.

$\Rightarrow$ G-bundles on the curve should realize LLC.

$\rightsquigarrow$ Try to "do" geometric Langlands on the FF curve.
see how it relates to LLC.

<u>Notations</u> Everything works for any nonarch local field; for simplicity, $\mathbb{Q}_p$.

- $\Gamma = \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \supset W_{\mathbb{Q}_p}$, $\ell \neq p$.
- $G/\mathbb{Q}_p$ conn red gp.
- $\bar{\mathbb{F}}_p$ alg closure of $\mathbb{F}_p$, $\overset{\sigma}{\curvearrowright}\breve{\mathbb{Q}}_p = W(\bar{\mathbb{F}}_p)[\frac{1}{p}]$
- $B(G) = G(\breve{\mathbb{Q}}_p) / \sigma\text{-conj}$.
- $\pi_1(G) := \pi_1(G_{\bar{\mathbb{Q}}_p})$ Borovoi
- $\underset{\Gamma}{\overset{\curvearrowleft}{X_*(T)}} = X_*(T_{\bar{\mathbb{Q}}_p}) \supset X_*(T)^+$.

universal Cartan

<u>Recall</u> $S = \mathrm{Spa}(R, R^+)$ perf'd space $/\mathbb{F}_p$ (usually $/\bar{\mathbb{F}}_p$)
R Tate & perfect ring.

$$Y_S = \mathrm{Spa}(W(R^+)) \setminus \{p[\varpi]=0\} \quad (/\breve{\mathbb{Q}}_p) \qquad X_S = Y_S/\phi^{\mathbb{Z}} \quad (/\mathbb{Q}_p).$$

$\circlearrowleft \phi$

Given any adic space $Z/\mathbb{Z}_p$, get v-sheaf

$$Z^{\diamond} : \mathrm{Perf}_{\mathbb{F}_p}^{op} \longrightarrow \mathrm{Sets}$$
$$S \longmapsto \{S^\#\text{ untilts of } S \ \&\ S^\# \to Z \text{ map}\}.$$

If Z analytic, then $Z^\diamond$ diamond = quotient of perf'd space by a proét equiv relation.

locally $\mathrm{Spa}(A,A^+)$, A Tate (i.e. contains top nilp unit).

Adic spaces $\supset$ analytic adic spaces

schemes $\subset$ formal schemes $\subset$ Adic spaces; rigid spaces, generic fibers of formal schemes $\subset$ analytic adic spaces.

E.g. (1) $Z = \mathrm{Spa}\,\mathbb{Q}_p = \mathrm{Spa}\,\mathbb{C}_p/\Gamma_{\mathbb{Q}_p}$

$$Z^\diamond = (\mathrm{Spa}\,\mathbb{C}_p)^\diamond/\Gamma_{\mathbb{Q}_p} = (\mathrm{Spa}\,\mathbb{C}_p^\flat)/\Gamma_{\mathbb{Q}_p}$$

$$\mathrm{Spa}\,\mathbb{Q}_p = \mathrm{Spa}\,\mathbb{Q}_p^{cycl}/\mathbb{Z}_p^\times = \mathrm{Spa}\,\mathbb{F}((t^{1/p^\infty}))/\mathbb{Z}_p^\times$$

$\mathbb{Q}_p^{cycl} = \mathbb{Q}_p(\zeta_{p^\infty})^\wedge$, $\gamma \in \mathbb{Z}_p^\times$

$t = \varepsilon - 1$, $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in (\mathbb{Q}_p^{cycl})^\flat$

$\gamma(t) = (1+t)^\gamma - 1$ defines the $\mathbb{Z}_p^\times$-action

(2) $(\mathbb{G}_{m,\mathbb{C}_p})^\diamond = \widetilde{\mathbb{G}}_{m,\mathbb{C}_p}^\diamond/\mathbb{Z}_p$, $\widetilde{\mathbb{G}}_m = \varprojlim_{x\mapsto x^p} \mathbb{G}_m$

perf'd

$$\widetilde{\mathbb{G}}_m \xrightarrow{\mathbb{Z}_p\text{-cover}} \mathbb{G}_m$$

$$\rightsquigarrow \widetilde{\mathbb{G}}_{m,\mathbb{C}_p^\flat}/\mathbb{Z}_p = \widetilde{\mathbb{G}}_{m,\mathbb{C}_p}^\diamond/\mathbb{Z}_p.$$

Prop'n (1) $|Z| \cong |Z^\diamond|$ if Z analytic. (Have $Z_{ét} \cong Z^\diamond_{ét}$)

(Z non-analytic: c.f. Ian Gleason.)

(2) $Z \to Z'$ univ homeomorph

$\Rightarrow \quad Z^\diamond \xrightarrow{\cong} Z'^\diamond$.

(3) (Kedlaya-Liu)

$$\left\{ \begin{matrix} \text{(semi) normal rigid} \\ \text{spaces } / \mathbb{Q}_p \end{matrix} \right\} \longrightarrow \{ \text{diamonds} / (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond \}$$

$$X / \mathbb{Q}_p \longmapsto X^\diamond / \mathbb{Q}_p^\diamond$$

is fully faithful.

$$Y_S^\diamond = S \times (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond \longrightarrow (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond.$$

$$\rightsquigarrow Y_S \to \mathrm{Spa}\, \mathbb{Q}_p.$$

Think $X_S^\diamond = S/\mathrm{Frob}_S \times (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond \longrightarrow (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond$

$$\rightsquigarrow X_S \to \mathrm{Spa}\, \mathbb{Q}_p.$$

If $S^\#$ untilt of S, get $(S^\#)^\diamond \longrightarrow S \times (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond \quad / (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond$

$\quad\quad (S^\#)^\diamond = S$

$$S^\# \xhookrightarrow[\text{divisor}]{\text{Cartier}} Y_S \quad / \mathrm{Spa}\, \mathbb{Q}_p$$

see Weinstein (2/2) for this.

From now on, will work on $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$.

__Def'n__ Bun_G is the v-stack $\quad S \longmapsto \{ G\text{-bundles on } X_S \}$

$\quad\quad S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$ — a groupoid.

Structure of Bun_G

(I) $S = \mathrm{Spa}\, C$,

C complete alg closed nonarch field $/ \overline{\mathbb{F}}_p$.

$\underline{\text{Thm}}$ (Fargues-Fontaine, Fargues, Anschütz)
$\qquad\qquad GL_n \qquad\qquad$ general G $\qquad\qquad \mathbb{F}_p((t))$

Set of G-isocrystals $\{\boxed{B(G)} \longrightarrow \mathrm{Bun}_G(C)/\cong$ bijection.

$$G(\breve{\mathbb{Q}}_p) \ni b \longmapsto \mathcal{E}_b = G \times Y_S / b \times \phi$$
$$\downarrow$$
$$Y_S/\phi = X_S.$$

$$\Rightarrow \quad |\mathrm{Bun}_G| = B(G).$$

(II) $B(G) \xrightarrow[\text{Kottwitz}]{(\nu,\kappa)} (X_*(T)^+_{\mathbb{Q}})^\Gamma \times \pi_1(G)_\Gamma$

ν : Newton pt, κ : Kottwitz invariant.

$\underline{\text{Thm}}$ (Kedlaya-Liu, Fargues-Scholze, Hansen, Viehmann)
$\qquad \nu \qquad\qquad \kappa \qquad\qquad$ (3) GL_n $\qquad$ (3) general

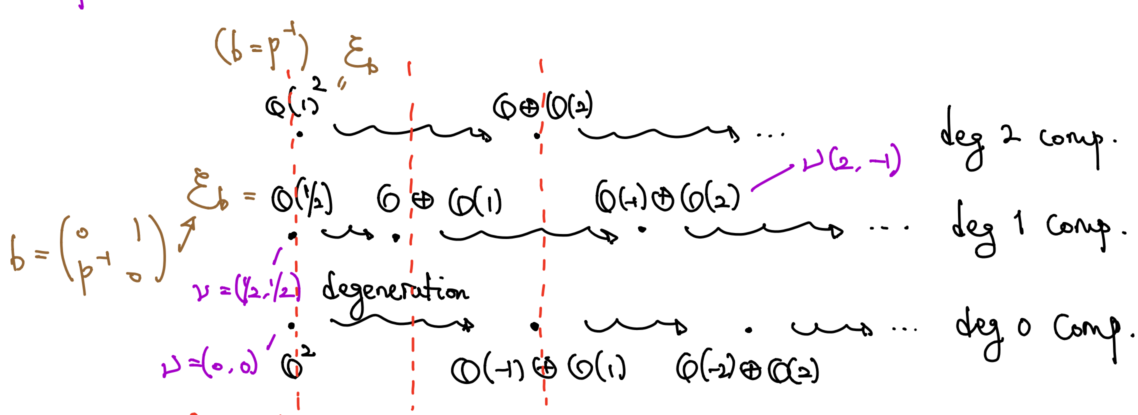
(1) ν semi continuous. $\left(|\mathrm{Bun}_G| = B(G) \xrightarrow{\nu} (X_*(T)^+_{\mathbb{Q}})^\Gamma.\right)$

(2) κ locally constant, $\pi_0 \mathrm{Bun}_G \xrightarrow[\kappa]{\sim} \pi_1(G)_\Gamma$

(3) $|\mathrm{Bun}_G| \xrightarrow[\text{homeo}]{\cong} B(G)$ w.r.t. order top

$(\nu,\kappa) \le (\nu',\kappa')$ if $\nu \le \nu'$ in dom order & $\kappa = \kappa'$.

$\underline{\text{Picture for } GL_2}$



strata of codim 0 $\quad$ codim 1 $\quad$ codim 2 $\quad \cdots \quad$ $\mathrm{codim}\, \mathrm{Bun}_G^b = \langle 2\rho, \nu(b) \rangle$.

κ = degree $\quad \pi_1(G) = \mathbb{Z}, \quad X_*(T) = \mathbb{Z}^2, \quad X_*(T)_{\mathbb{Q}} = \mathbb{Q}^2,$

$\circlearrowleft$ Γ trivial $\qquad X_*(T)^+_{\mathbb{Q}} = \{(\lambda_1, \lambda_2) : \lambda_1 \geq \lambda_2\}.$

(III) Each connected comp has a $\underbrace{\text{unique semistable pt}}$

$\updownarrow$

$b \in B(G)$ basic

with $B(G)_{\text{basic}} \xrightarrow[\kappa]{\cong} \pi_1(G)_\Gamma$.

$\mathrm{Bun}_G^{ss} \underset{\text{open}}{\subseteq} \mathrm{Bun}_G$ semistable locus.

Thm $\mathrm{Bun}_G^{ss} = \coprod_{\substack{b \in B(G)_{\text{basic}} \\ \kappa \downarrow \wr \\ \pi_1(G)_\Gamma}} \mathrm{Bun}_G^b$ where $\mathrm{Bun}_G^b = [*/\underline{G_b(\mathbb{Q}_p)}]$, $\underline{G_b(\mathbb{Q}_p)} = \underline{\mathrm{Aut}}(\mathcal{E}_b)$.

When $b = 1$, $G_b = G$ & $\mathrm{Bun}_G^1 = [*/\underline{G(\mathbb{Q}_p)}]$.

Generally, $\quad G_b \times_{\mathbb{Q}_p} X_S = \underline{\mathrm{Aut}}_{X_S}(\mathcal{E}_b)$

$\uparrow$ inner form of G $\quad \uparrow$ inner form of G over X_S.

$\mathrm{Aut}(\mathcal{E}_b) = H^0(X_S, \underline{\mathrm{Aut}}_{X_S}(\mathcal{E}_b)) = \underline{G_b(\mathbb{Q}_p)}(S)$

In particular, $\mathrm{Rep}(G_b(\mathbb{Q}_p)) = \mathrm{Shv}([*/\underline{G_b(\mathbb{Q}_p)}]) \subseteq \mathrm{Shv}(\mathrm{Bun}_G)$

$\qquad \mathrm{Shv}(\mathrm{Bun}_G^b)$ $\hookrightarrow$ ext'n by zero

(IV) General b.

$[*/\mathcal{G}_b] = \mathrm{Bun}_G^b \subseteq \mathrm{Bun}_G$ locally closed.

$\mathcal{G}_b = \mathrm{Aut}(\mathcal{E}_b)$.

$$1 \longrightarrow \mathcal{G}_b^\circ \longrightarrow \mathcal{G}_b \longrightarrow \underbrace{\underline{G_b(\mathbb{Q}_p)}} \longrightarrow 1$$

Connected "unipotent" $\qquad$ σ-centralizer of b

(G_b inner form of a Levi of G (if G qsplit))

exact seq of sheaves of groups on $\mathrm{Perf}_{\mathbb{F}_p}$.

But $\mathcal{G}_b^\circ$ cannot act nontrivially on (ℓ-adic) sheaves

$$\begin{aligned}\mathrm{Rep}(G_b(\mathbb{Q}_p)) &= \mathrm{Shv}([*/G_b(\mathbb{Q}_p)]) \\ &= \mathrm{Shv}([*/\mathcal{G}_b]) = \mathrm{Shv}(\mathrm{Bun}_G^b) \\ &\subseteq \mathrm{Shv}(\mathrm{Bun}_G).\end{aligned}$$

$\Rightarrow$ $\mathrm{Shv}(\mathrm{Bun}_G)$ is glued from all $\mathrm{Shv}(\mathrm{Bun}_G^b) \simeq \mathrm{Rep}(G_b(\mathbb{Q}_p))$.

<u>Example</u> $b = \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix} \rightsquigarrow \mathcal{E}_b \simeq \mathcal{O}(1) \oplus \mathcal{O}$

$$\mathrm{Aut}(\mathcal{E}_b) = \mathrm{Aut}(\mathcal{O}(1) \oplus \mathcal{O}) = \begin{pmatrix} \mathbb{Q}_p^\times & \boxed{H^0(\mathcal{O}(1))} \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}$$

$\boxed{H^0(\mathcal{O}(1))} = \mathcal{G}_b^\circ$

$\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times = G_b(\mathbb{Q}_p)$. $G_b = \mathbb{G}_m^2 = T$.

$$\begin{aligned} H^0(\mathcal{O}(1)) &\simeq \text{perf'd open unit disc} \\ &= \{ x \in R \mid |x-1| < 1 \} \end{aligned}$$

$\log[x] \leftarrow x$

$\dim \mathcal{G}_b = \dim \mathcal{G}_b^\circ = \langle 2\rho, \nu(b) \rangle$

$\dim \mathrm{Bun}_G^b = -\langle 2\rho, \nu(b) \rangle$.

(v) Bun_G "smooth Artin v-stack of dim 0"

$\quad \cup\!\mathsf{I}$

Bun_G^b "smooth Artin v-stack of dim $-\langle 2\rho, \nu(b)\rangle$".

§2 Banach-Colmez spaces

Fix $C/\mathbb{Q}_p$. Work on $\mathrm{Perf}_{C^\flat}$.

<u>Def'n</u> The category of Banach-Colmez spaces is the subcat of sheaves of $\mathbb{Q}_p$-v.s. on $\mathrm{Perf}_{C^\flat}$ generated by $\underline{\mathbb{Q}_p}$, $(\mathbb{A}_C^1)^\diamond$ $(\mathbb{Q}_p, C)$. (under direct sum, ext'n, quotients). (Colmez, Le Bras)

Prop If $\mathcal{E}$ coherent sheaf on X_C,

$$S \longmapsto H^0(X_S, \mathcal{E}|_{X_S})$$
$$S \longmapsto H^1(X_S, \mathcal{E}|_{X_S})$$

are Banach-Colmez spaces.

Thm (Le Bras) $\exists$ derived equiv

$$D^b\,\mathrm{Coh}(X_C) \simeq D^b(\text{BC spaces})$$

Examples (1)
$$\begin{tikzcd} \mathrm{Spa}\, C \arrow[r, hook, "i"] & X_C \\ S^\# \arrow[u] \arrow[r, hook] & X_S \arrow[u] \end{tikzcd}$$

$i_* C$ coh sheaf on X_C

$H^0(X_S, i_*C|_{X_S}) = \mathcal{O}(S^\#)$

rep'd by $(\mathbb{A}^1_C)^\diamond$.

(2) $\mathcal{E} = \mathcal{O}$, $H^0(X_C, \mathcal{O}) = \mathbb{Q}_p$, $H^0(X_S, \mathcal{O}) = \underline{\mathbb{Q}_p}(S) = \mathrm{Cont}(|S|, \mathbb{Q}_p)$

(3) $\mathcal{E} = \mathcal{O}(1)$, $0 \longrightarrow \mathcal{O} \xrightarrow{t} \mathcal{O}(1) \longrightarrow i_*C \longrightarrow 0$ $\quad t = \log[\varepsilon]$,

perf'd unit disc

$$0 \longrightarrow \underline{\mathbb{Q}_p} \longrightarrow H^0(\mathcal{O}(1)) \longrightarrow (\mathbb{A}^1_C)^\diamond \longrightarrow 0$$
$$x \longmapsto \log x^\#$$

(4) $\mathcal{E} = \mathcal{O}(-1)$, $0 \longrightarrow \mathcal{O}(-1) \xrightarrow{t} \mathcal{O} \longrightarrow i_*C \longrightarrow 0$.

$$0 \longrightarrow \underline{\mathbb{Q}_p}^\diamond \longrightarrow (\mathbb{A}^1_C)^\diamond \longrightarrow H^1(X_C, \mathcal{O}(-1)) \longrightarrow 0$$
$$H^1(X_C, \mathcal{O}(-1)) = (\mathbb{A}^1_C)^\diamond / \underline{\mathbb{Q}_p}.$$

(5) $0 \longrightarrow i_*C \xrightarrow{t} \mathcal{O}/t^2 \longrightarrow i_*C \longrightarrow 0$

$$\rightsquigarrow 0 \longrightarrow (\mathbb{A}^1_C)^\diamond \longrightarrow \underbrace{H^0(\mathcal{O}/t^2)}_{\text{not a rigid space}} \longrightarrow (\mathbb{A}^1_C)^\diamond \longrightarrow 0 \quad \text{highly nonsplit}$$