Lecture 1. Construction of automorphic Galois representations

(Three lectures on "Shimura varieties and Modularity")

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Plan

Part I. Main Theorem on "automorphic $\rightsquigarrow$ Galois"

- ℓ-adic coefficients
- **torsion** coefficients

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Part IV. Construction of torsion Galois representations

- by p-**adic congruences** à la Scholze, based on Parts II and III.

Part I. Main Players

Fix a number field F, a prime ℓ, and $n \in \mathbb{Z}_{\geq 1}$.

S : a finite set of places of $F \supset \{\text{places} \,|\, \ell, \infty\}$,

$K^S = \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_{F_v}) \subset \mathrm{GL}_n(\mathbb{A}_F^S)$.

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Galois side with coeff. $k \in \{\bar{\mathbb{Q}}_\ell, \bar{\mathbb{F}}_\ell\}$

$\boxed{\mathcal{G}^S(n,F)_k}$:= continuous semisimple unramified-outside-S reps

$$\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(k).$$

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Remark

- $\mathrm{GLC}_{\bar{\mathbb{Q}}_\ell,\iota}$ and $\mathrm{GLC}_{\bar{\mathbb{F}}_\ell}$ are uniquely characterized. Images?
- ① $\not\Rightarrow$ ②.
- $\exists$ an upgrade $\mathfrak{m} \mapsto \rho_{\mathfrak{m}}$ lifting ②, where $\rho_{\mathfrak{m}}$ has coeff. in a Hecke algebra. (Caraiani's talk)
- $\exists$ conjecture for general G (Buzzard–Gee).

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Theorem (Harris–Lan–Taylor–Thorne, Scholze)

If F is a totally real or CM field,

- *① is true on $\pi \in \mathcal{A}^S_{\mathrm{rac}}(n,F)$.*
- *② is true.*

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Define $\widetilde{\mathcal{A}}^S_{\mathrm{rac}}(n,F) := \{\pi \circ c \simeq \pi^\vee\} \subset \mathcal{A}^S_{\mathrm{rac}}(n,F)$ "conjugate self-dual"

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Plan: Briefly go over #1 (Part II), focus on #2 (Parts III, IV).

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Theorem (Clozel, Kottwitz, Harris–Taylor, ...)

If F is a totally real or CM field, then

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- F is CM. Set $F^+ :=$ fixed field of $c \curvearrowright F$.
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(ii) Langlands–Kottwitz method to show "s.t. ...", cf. Morel's lectures.

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Outline

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(i) "automorphic descent" (inverse of base change, not unique).

(ii) Langlands–Kottwitz method to show "s.t. ...", cf. Morel's lectures.

(iii) Ramanujan conjecture for Π as a by-product.

Part II. Conjugate self-dual case: reality

$U/F^+ :=$ unitary group in n variables w.r.t. F/F^+ s.t.

1. signature at ∞ is $(1, n-1), (0, n), ..., (0, n)$.
2. U is q-split at all finite places.

Problem

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- Reduce via congruences to older results by Clozel and Kottwitz where ② is given up (Fintzen–S.–Beuzart-Plessis)

Coming up next

Good news: You can forget almost everything so far.

Part III: Perfectoid Shimura varieties

- Anti-canonical tower
- **Hodge–Tate period morphism**

Part IV: Construction of torsion Galois reps

- Obstruction: locally sym spaces for GL_n are **not** Shimura varieties (also see Johansson–Thorne)
  $\rightsquigarrow$ pass to Shimura variety for Sp_{2n} or U_{2n} via Borel–Serre + ...
- Comparison theorems ($\rightsquigarrow$ Čech cohomology)
- Fake Hasse invariants