Shimura varieties and modularity (1/3)

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(See slides for the first half.)

§1 Main theorem on "autom ⇝ Gal"

§2 (Conjugate) Self-dual case with ℓ-adic coeff

§3 Perfectoid Shimura varieties

(G, X) Hodge-type Shimura datum.

$$Sh^{*}_{K^p} = \varprojlim_{K_p} Sh^{*}_{K^pK_p}, \qquad K^pK_p \subseteq G(\mathbb{A}^\infty)$$

"p^∞-level" $(*)$ = min compactification.

$$\rightsquigarrow \quad \mathcal{S}^*_{K^p} \sim \varprojlim_{K_p} \mathcal{S}^*_{K^pK_p} \quad \text{adic space} \quad /\mathbb{C}_p, \quad (\mathbb{C}_p \simeq \mathbb{C}).$$

in sense of diamond lim.

$(G, X) \rightsquigarrow \mu : \mathbb{G}_m \to G$ cochar

$\rightsquigarrow \mathcal{F}\ell_\mu := G/P_\mu$ flag var

where P_μ = stabilizer of "μ-fil".

Siegel $\mathcal{F}\ell_\mu$ parametrizes all max isotropic subspaces.

Thm (Scholze, Scholze–Caraiani)

(1) $\mathcal{S}^*_{K^p}$ is perfectoid as K^p varies
$\circlearrowleft G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p)$

(2) $\exists$ Hodge-Tate morphism

$$\begin{array}{ccc} \pi_{HT} : \mathcal{S}^*_{K^p} & \longrightarrow & \mathcal{F}\ell_\mu \\ \circlearrowleft & & \circlearrowleft \\ G(\mathbb{A}^\infty) & \xrightarrow{\text{equiv}} & G(\mathbb{Q}_p) \end{array}$$

s.f. on $\mathbb{C}_p$-pts $(A, \cdots) \longmapsto$ HT-fil $\mathrm{Lie}\, A(1) \subset T_p A \otimes_{\mathbb{Z}_p} \mathbb{C}_p$.

(Diao-Lan-Liu-Zhu in general.)

<u>Toy example</u> $\varprojlim_{S \mapsto S^p} \mathrm{Spa}(\mathbb{C}_p\langle S\rangle, \mathcal{O}_{\mathbb{C}_p}\langle S\rangle)$ perfectoid (unit disc)

$= \mathrm{Spa}(\mathbb{C}_p\langle T^{1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{1/p^\infty}\rangle)$

$\downarrow$

$\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle)$ closed unit disc

<u>Key</u> (trans maps = "rel Frob") $\Rightarrow$ perfectoid lim (by def'n of perf'dness.)

<u>Proof</u> (Seigel) $G = GSp_{2n}$.

- Ignore boundary.
- $G(\mathbb{Z}_p) \supseteq \Gamma_0(p^m) \supseteq \Gamma(p^m)$ $(m \geq 0)$

$\begin{pmatrix} * & * \\ & * \end{pmatrix} \supseteq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ mod p^m

Drop the subscript K^p:

$S^*_{K^p} = S^*_{\Gamma(p^\infty)} = \varprojlim_m S^*_{\Gamma(p^m)}$

$$\begin{array}{l}
S^*_{\Gamma(p^\infty)} \\
\downarrow \\
S^*_{\Gamma_0(p^\infty)} \qquad \text{locus where } |\mathrm{Ha}| \geq |p|^\varepsilon, \quad 0 < \varepsilon < \tfrac{1}{2}. \\
\downarrow \qquad\quad \downarrow \qquad\qquad (\mathrm{Ha} = \text{Hasse inv, measuring ordinarity}) \\
S^*_{\Gamma_0(1)} \underset{\text{open}}{\supseteq} S^*_{\Gamma_0(1)}(\varepsilon) \underset{\text{strict}}{\supsetneq} S^*_{\Gamma_0(1)}(0) \quad \text{(allowing a little ss locus)} \\
\qquad\qquad \text{affinoid} \qquad\qquad \text{ordinary}
\end{array}$$

$$\rightsquigarrow \quad \begin{array}{ccc}
S^*_{\Gamma_0(p^\infty)} & \supseteq & \boxed{S^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\mathrm{anti}}} \leftarrow \text{desiderata} \\
\downarrow & & \downarrow f_0 \\
S^*_{\Gamma_0(1)} & \underset{\text{open}}{\supseteq} & S^*_{\Gamma_0(1)}(\varepsilon)
\end{array}$$

Step 1 Anti-canonical tower.

$\Gamma_0(p^m)$-level : $\underset{\substack{\Uparrow \\ \text{max isotropic}}}{D_m} \subseteq A[p^m]$.

$S^*_{\Gamma_0(p^m)}(\varepsilon)_{\text{anti}}$ = locus where $D_m \cap C_m = \{0\}$

(if ε small, $\exists!$ C_m canonical subgrp of $A[p^m]$)

Upshot Transition maps in $f_0 \equiv$ rel Frob mod $p^{1-\varepsilon}$.

$\Rightarrow$ $S^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\text{anti}}$ is perfectoid.

Can get level up :

$$\begin{tikzcd}
S^*_{\Gamma(p^\infty)} \arrow[d] \arrow[r, phantom, "\supseteq"] \arrow[dr, phantom, "\square"] & S^*_{\Gamma(p^\infty)}(\varepsilon)_{\text{anti}} \arrow[d, "f"] \\
S^*_{\Gamma_0(p^\infty)} \arrow[d] \arrow[r, phantom, "\supseteq"] & S^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\text{anti}} . \arrow[d, "f_0"] \\
S^*_{\Gamma_0(1)} \arrow[r, phantom, "\supseteq"] & S^*_{\Gamma_0(1)}(\varepsilon)
\end{tikzcd}$$

both affinoid perf'd.

Step 2 Purity : the fin ét over perf'd is perf'd

f is lim of fin étale (away from boundary)

$\Rightarrow$ $S^*_{\Gamma_0(p^\infty)}(\varepsilon)_{\text{anti}}$ is aff'd perf'd.

Step 3 Construct top HT map

$|\pi_{HT}| : |S^*_{K^p}| \xrightarrow{\text{equiv}} |\mathcal{F}\ell_\mu|$

$(A, \cdots) \longmapsto$ HT-fil $\quad$ $T_pA \otimes_{\mathbb{Z}_p} \mathbb{C}_p \underset{\substack{\Uparrow \\ \text{use } p^\infty\text{-level (at } p)}}{\simeq} \mathbb{C}_p^{2n}$

Top argument $\Rightarrow$ $\underset{\substack{\Uparrow \\ \text{need HT-fils in families.}}}{\text{continuous}}$.

Step 4 Propagate perf'dness. (Fix $0 < \varepsilon < \frac{1}{2}$).

$|\pi_{HT}| : |S_{K^p}| \longrightarrow |\mathcal{F}\ell_\mu|$

$|S^{\text{ord}}_{K^p}| \xrightarrow{\text{fact}} \mathcal{F}\ell_\mu(\mathbb{Q}_p)$

$\rightsquigarrow$ $\exists$ open nbhd $\mathcal{U}$ s.t. $\mathcal{F}\ell_\mu(\mathbb{Q}_p) \subsetneq \mathcal{U} \subseteq |\mathcal{F}\ell_\mu|$
s.t. $|\pi_{HT}|^{-1}(\mathcal{U}) \subseteq |S_{K^p}(\varepsilon)|$.

Can work explicitly to see

$$G(\mathbb{Q}_p) \cdot \mathcal{U} = |\mathcal{F}\ell_\mu|$$

$$\Rightarrow G(\mathbb{Q}_p) \cdot \underbrace{|S_{K^p}(\varepsilon)|}_{= G(\mathbb{Z}_p) \cdot |S_{K^p}(\varepsilon)_{\text{anti}}|} = |S_{K^p}|$$

b/c $G(\mathbb{Z}_p)$-action "permutes" all possible max isotropics.

$$\Rightarrow G(\mathbb{Q}_p) \cdot |S_{K^p}(\varepsilon)_{\text{anti}}| = |S_{K^p}| \leftarrow \text{perf'd.}$$

<u>Step 5</u> Upgrade $|\pi_{HT}|$ to $\pi_{HT} : S^*_{K^p} \to \mathcal{F}\ell_\mu$.
(recycle Step 3).

<u>§4 Constructing torsion Galois rep'ns</u>

Given $\mathfrak{m} \in \mathcal{HE}^S(n, F)$.

<u>Want</u> $\mathfrak{m}$ is mod p of "$\underbrace{\text{classical } \mathcal{HE}}_{\text{some Hecke eigenchar appearing in cohom of Sh vars with char 0 coeff.}}$" (up to "doubling")

$\uparrow$ really hard

easy $\left\{ \begin{array}{l} \rightsquigarrow \rho : \mathrm{Gal}_F \to GL_n(\overline{\mathbb{Q}}_\ell) \\ \overset{\text{mod } p}{\rightsquigarrow} \bar\rho_{\mathfrak{m}} \oplus \bar\rho_{\mathfrak{m}}^\vee + \text{some 1-dim'l to } GL_n(\overline{\mathbb{F}}_\ell). \end{array} \right.$