The local Langlands conjecture (1/3)

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<u>Goal</u> state local Langlands conjectures.

<u>Plan</u> (1) LLConjs (smooth irreps of $G(F) \to$ Langlands parameters)

(2) refined LLCs (about fibers of this map) for quasi-split groups

(3) refined LLC in general (via Galois gerbs).

§1 Smooth rep'n of reductive groups

<u>Notation</u> F local field, $\|\cdot\| : F^\times \to \mathbb{R}_{>0}$ non-arch,

$\|\varpi_F\|^{-1} = q =$ card of residue field, $\varpi_F =$ uniformizer.

C alg closed field of char 0 (e.g. $C = \mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$)

<u>Smooth rep'ns</u> G conn reductive grp $/F$

(e.g. $G = GL_n, SL_n, Sp_{2n}, Spin_{2n+1}, E_8, \dots$).

⇝ sm rep'n $\pi : G(F) \to GL(V)$, $V = C$-v.s. $\circlearrowleft G(F)$

with $G(F) \times V \to V$ conti. for the discrete top on V.

<u>Fact</u> (V, π) irr smooth rep'n $\Rightarrow$ admissible

($\forall K \leq G(F)$ compact subgrp, $\dim_C V^K < \infty$)

When F is archimedean: $(\mathfrak{g}, K)$-modules instead of sm rep'n:

$\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\, G(F)$, $K =$ max'l compact subgrp of $G(F)$.

More notations irrep'n (V,π)

$\rightsquigarrow$ central character $\omega_\pi : Z(G(F)) \to \mathbb{C}^\times$

& Contragredient rep'n $(\tilde{V}, \tilde{\pi})$.

Algebraic notions (1) Parabolic induction: $P = MN$ parabolic subgrp of G (Levi decomp.)

(V,σ) smooth rep'n of $M(F)$

$$\rightsquigarrow i_P^G(\sigma) = \left\{ f : G(F) \to GL(V) \;\middle|\; \begin{array}{l} f \text{ sm}, \ f(pg) = \delta_P^{1/2}(p)\sigma(p) f(g) \\ \forall p \in P(F),\ g \in G(F) \end{array} \right\}$$

note $\underbrace{\delta_P^{1/2}(p)}_{q}$: for preserving the unitarizability.

need to choose $\sqrt{q} \in C$ (if $C = \mathbb{C}$, $\sqrt{q} > 0$).

(2) Jacquet functor: (V,π) sm rep'n of $G(F)$, $P = MN$.

$\Rightarrow V_N$ = coinvariants under $N(F)$ $\circlearrowleft_{\pi_N} M(F)$

$\uparrow$ (quotient)

V $\quad\rightsquigarrow r_P^G(\pi) := \delta_P^{-1/2} \otimes \pi_N$.

Def'n (V,π) irrep is <u>supercuspidal</u> iff $\forall P$, $V_N = 0$.

Equivalently, if the matrix coefficients

$$G(F) \longrightarrow \mathbb{C},\ g \mapsto \langle \pi(g)v, \tilde{v} \rangle,\ v \in V,\ \tilde{v} \in \tilde{V}$$

having compact supports mod $Z(G(F))$.

Rough classification of irrep'ns of $G(F)$ by their supercuspidal supports:

Theorem (1) Any irrep'n π of $G(F)$ embeds in some $i_P^G(\sigma)$

σ = supercuspidal irrep of $M(F)$.

(2) If π occurs as a subquotient of some $i_{P'}^G(\sigma')$, (σ' supercusp)

then $(M,\sigma) \sim (M',\sigma')$ via $G(F)$-conj.

Asymptotic (topological) notions

$C = \mathbb{C}$, (V, π) irrep of $G(F)$.

* If ω_π is unitary, say that π is essentially square-integrable if $\forall v \in V$, $\tilde{v} \in \tilde{V}$, (aka. ess. L^2)

$$\int_{G(F)/Z(G(F))} |\langle \pi(g) v, \tilde{v} \rangle|^2 \, dg < \infty.$$

($\Rightarrow$ π embeds in $L^2(G(F), \omega_\pi)$.)

* In general, $\exists!$ $\chi : G(F) \longrightarrow \mathbb{R}_{>0}$ cont's char s.t. $\chi \otimes \pi$ has unitary central char $\omega_{\chi\otimes\pi}$.

Say that π is ess L^2 if $\chi \otimes \pi$ is.

This property can be checked on Jacquet mods.

M Levi of G. A_M max'l central split torus in M.

$$\mathfrak{a}_M^* = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}.$$

$$\rightsquigarrow \quad \mathfrak{a}_M^* \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cont}}(A_M(F), \mathbb{R}_{>0})$$

$$\chi \otimes s \longmapsto (x \longmapsto \|\chi(x)\|^s).$$

Prop'n (V, π) irrep of $G(F)$. Assume ω_π unitary.

$$\pi \text{ ess } L^2 \iff \begin{pmatrix} \forall P = MN,\ \forall \chi : A_M(F) \to \mathbb{C}^\times \text{ occuring in } r_P^G(\pi), \\ |\chi| \text{ is a linear combination with positive coeffs} \\ \text{of the simple roots of } A_M \text{ acting on } N. \end{pmatrix}.$$

Def'n (V, π) irrep is *tempered* if the condition above holds with "nonnegtive" instead of "positive".

Rmk Tempered rep'ns are the ones occuring in "the" Plancherel formula

$$f(1) = \int_{\text{irreps } \pi} \mathrm{tr}(\pi(f)) \, d\mu(\pi).$$

If ω_π is unitary, then

scusp $\Rightarrow$ ess L^2 $\Rightarrow$ tempered $\Rightarrow$ unitary.

"Classification" of tempered in terms of ess L^2 irreps of Levi's.

Prop (1) $P = MN$, σ ess L^2 irrep of $M(F)$, ω_σ unitary.

Then $i_P^G(\sigma)$ is semi-simple, has fin length, and its any constitued is tempered.

(2) (P, σ) & (P', σ') as in (1). Then

$i_P^G(\sigma)$ & $i_{P'}^G(\sigma')$ have an irred subrep

$\Leftrightarrow$ $(M, \sigma) \sim (M', \sigma')$ via $G(F)$-conj

If it is the case, then $i_P^G \sigma \simeq i_{P'}^G \sigma'$.

(3) Any tempered irrep π of $G(F)$ occurs in some $i_P^G(\sigma)$ as in (1).

In general, these $i_P^G(\sigma)$ are not irred.

Langlands' classification

(Langlands / $\mathbb{R}$, Silberger / nonarch).

Thm (1) $P = MN$, σ tempered irrep of $M(F)$,

$\nu : M(F) \to \mathbb{R}_{>0}$ cont char.

Under a certain positivity condition on ν,

$i_P^G(\sigma \otimes \nu)$ has an irred quotient rep $J(P, \sigma)$

(also unique irr subrep of $i_{\bar{P}}^G(\sigma \otimes \nu)$.)

(2) $\forall$ irrep π of $G(F)$, $\exists !\, (P, \sigma, \nu)$ (up to $G(F)$-conj)

s.t. $\pi \simeq J(P, \sigma, \nu)$.

Harish-Chandra char

Fix Haar measure on $G(F)$.

$$C_c^\infty(G(F)) = \{\text{sm compactly supported functions } f : G(F) \to \mathbb{C}\}.$$

By admissibility, $\pi(f) : V \longrightarrow V$, $\pi(f)v = \int_{G(F)} f(g)\cdot \pi(g) v\, dg$.

has finite range

V^K for some K

$\rightsquigarrow \operatorname{tr}(\pi(f)) =: \Theta_\pi(f)$

$\rightsquigarrow \Theta_\pi : C_c^\infty(G(F)) \longrightarrow \mathbb{C}$ determines a fin length π up to semisimplification.

<u>Thm</u> (Harish-Chandra) $F/\mathbb{Q}_p$ fin ext'n.

$\exists!\ \Theta_\pi \in L^1_{loc}(G(F))$ s.t. $\forall f \in C_c^\infty(G(F))$,

$$\Theta_\pi(f) = \int_{G(F)} f(g)\cdot \Theta_\pi(g)\, dg$$

& Θ_π is invariant under conj by $G(F)$.

rep'd by a unique sm fcn on $G_{rs}(F)$

reg semisimple locus of $G(F)$

<u>note</u> $\mathrm{vol}(G(F)\setminus G_{rs}(F)) = 0$.

§2 Langlands dual groups

$\bar{F}/F$ sep closure, $\Gamma = \mathrm{Gal}(\bar{F}/F)$.

Based root data: $\exists$ fin subext'n E/F of $\bar{F}/F$

& Killing (Borel) pair (B,T) in G_E.

$\rightsquigarrow$ based root datum $(X, R, R^\vee, \Delta)$

- $X = X^*(T)$, $R = (\text{roots of } T \text{ in } G_E) \subset X$

$R^\vee \subset X^\vee = \mathrm{Hom}(X,\mathbb{Z}) = X_*(T)$ coroots

· $\Delta \subset R$ simple roots for B.

This is canonical & has sm action of Γ.

Functor brd_F : gpoid of conn red gps $\longrightarrow$ based root data $\circlearrowleft \Gamma$.

<u>Def'n</u> Groupoid $IT(G)$ of inner twists of G

· objects (G', ψ)

where $\psi : G_{\bar F} \xrightarrow{\sim} G'_{\bar F}$ s.t. $\forall \sigma \in \Gamma$, $\psi^{-1}\sigma(\psi) \in G_{ad}(\bar F) = \mathrm{Inn}(G_{\bar F})$.

· morphs $\mathrm{Hom}_{IT(G)}((G_1, \psi_1), (G_2, \psi_2))$

$= \{ g \in G_{ad}(\bar F) \mid \psi_2^{-1}\sigma(\psi_2) = \mathrm{Ad}(g) \cdot \psi_1^{-1}\sigma(\psi_1) \cdot \mathrm{Ad}(\sigma(g))^{-1} \}$.

<u>Rmks</u> (1) $(G', \psi) \in \mathrm{Ob}(IT(G)) \Rightarrow \mathrm{brd}_F(G) \simeq \mathrm{brd}_F(G')$.

(2) $\Gamma \longrightarrow G_{ad}(\bar F)$, $\sigma \longmapsto \psi^{-1}\sigma(\psi)$

(3) Hom in $IT(G)$ $\rightsquigarrow$ $\psi_2 \mathrm{Ad}(g) \psi_1^{-1} : G_{1,\bar F} \xrightarrow{\sim} G_{2,\bar F}$ is def'd over F.

(4) $\mathrm{Aut}_{IT(G)}(G', \psi) = G'_{ad}(F)$.

<u>Prop</u> b brd $\circlearrowleft \Gamma$, CRG_b groupoid of pairs (G, α) (conn red gps)

where G conn red $/F$, $\alpha : b \simeq \mathrm{brd}_F(G)$.

(1) $\exists !$ (up to isom) $(G^*, \alpha^*) \in \mathrm{Ob}(CRG_b)$

s.t. G^* is quasi-split (i.e. has a Borel subgp $/F$)

(2) Choose $(G, \alpha) \in \mathrm{Ob}(CRG_b)$. Get an equiv:

$$Z^1(F, G_{ad}) \xleftarrow{\sim} IT(G) \xrightarrow{\sim} CRG_b.$$

<u>Langlands dual gps</u>

$G \rightsquigarrow \mathrm{brd}_F(G) = (X, R, R^\vee, \Delta) \circlearrowleft \Gamma$

Take dual brd $(X^\vee, R^\vee, R, \Delta^\vee)$

$\rightsquigarrow$ form pinned conn red gp $(\hat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ over $\mathbb{C}$.

Pinning splits:

$$1 \longrightarrow \underset{\hat{G}_{ad}}{\underset{\parallel}{\mathrm{Inn}(\hat{G})}} \longrightarrow \mathrm{Aut}(\hat{G}) \longrightarrow \mathrm{Out}(\hat{G}) \longrightarrow 1$$

section given by pinning

So Γ acts on $(\hat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$

$\rightsquigarrow$ naturally, take ${}^L G := \hat{G} \rtimes \Gamma$.

E.g.

G	GL_n	SL_n	$SO_{2n+1}^{?}$	$Spin_{2n}^{?}$
$\hat{G}$	GL_n	PGL_n	Sp_{2n}	PSO_{2n}

<u>Prop'n</u> · G ss simply conn $\Leftrightarrow$ $\hat{G}$ adjoint (i.e. $\hat{G} = \hat{G}_{ad}$).

· G_{der} s.c. $\Leftrightarrow$ $Z(\hat{G})$ is a torus $(\simeq \widehat{G/G_{der}})$

· ${}^L G_1 \simeq {}^L G_2 \Leftrightarrow G_1, G_2$ are inner forms of each other.

<u>Functoriality</u> $G = G_1 \times G_2$, ${}^L G \simeq {}^L G_1 \times_\Gamma {}^L G_2$.

Central isogeny $\theta : G \to H$ induces ${}^L\theta : {}^L H \to {}^L G$.

Given T max torus of G. Choose B Borel subgp of $G_{\bar{F}}$ containing $T_{\bar{F}}$.

$\rightsquigarrow$ get $\hat{T} \simeq \mathcal{T}$ $\underbrace{\text{not } \Gamma\text{-equiv}}$

Γ actions differed by a 1-cocycle $\Gamma \to$ Weyl gp

$\rightsquigarrow Z(\hat{G}) \hookrightarrow \hat{T}$ Γ-equiv.