Local shtukas and the Langlands program (1/2)

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§1 Review of Drinfeld's shtukas

X curve / $\mathbb{F}_q$ proj, nonsing, geom conn.

$K = k(X)$ function field.

Goal construct Langlands corr for GL_n / K.

Method Cohomology of space of shtukas.

Inspiration Autom forms on GL_n / K

(fcn on $G(K)\backslash G(\mathbb{A}) / \prod_v G(\mathcal{O}_v) = |\mathrm{Bun}_G(\mathbb{F}_q)|$, $G = GL_n$).

Let $S/\mathbb{F}_q$ be a scheme.

Def'n An X-shtuka over S of rk n is: (by Drinfeld)

- $x_1, x_2 : S \to X$ "legs"
- $\mathcal{E} \in \mathrm{Bun}_n(S)$ (= v.b. on $X \times_{\mathbb{F}_q} S$)
- $f : \mathrm{Fr}_S^* \mathcal{E} \dashrightarrow \mathcal{E}$ isom away from Γ_{x_i} $(i=1,2)$ with simple zero at x_1 & simple pole at x_2.

where $\Gamma_{x_i} \subset X \times S$ graph of x_i along $S \to X$.

Moduli stack of shtukas

$$\mathrm{Sht}^2(GL_n, \mu) \longrightarrow X^2, \quad \mu = (\mu_1, \mu_2)$$

$$(x_1, x_2, \mathcal{E}, f) \longmapsto (x_1, x_2)$$

to test on what extent f fails to be an isom (globally)

$\mu_1 = (1, 0, \cdots, 0)$ simple zero

$\mu_2 = (0, \cdots, 0, -1)$ simple pole.

Add level structure at

$\mathcal{U} \subseteq G(\mathbb{A})$ compact open

$$\mathrm{Sht}^2(GL_n, \mu)_{\mathcal{U}} \longrightarrow \eta_2 \in X^2 \text{ generic pt.}$$

Outcome (Drinfeld / Lafforgue):

$$\varinjlim_{\mathcal{U}} H^*(\mathrm{Sht}^2(GL_n,\mu)_{\mathcal{U},\bar{\eta}^2}, \bar{\mathbb{Q}}_\ell)$$

$$\circlearrowleft \; G(\mathbb{A}) \times \mathrm{Gal}(\bar{K}/K)^2$$

$$\simeq \bigoplus_{\substack{\pi \text{ cusp autom} \\ \text{rep of } G/K}} \pi \otimes \phi_\pi \otimes \phi_\pi^\vee$$

__Dream__ Define shtukas over number fields.

__Problem__ "$\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$" unknown.

it is $\mathrm{Spec}\,\mathbb{Z}$ as sch, but cannot carry $\mathrm{Gal}(\bar{K}/K)^2$.

Today: try to define "$\mathrm{Spec}\,\mathbb{Z}_p \times \mathrm{Spec}\,\mathbb{Z}_p$".

§2 Fontaine's $A_{\inf}$

Attempt to define "$\mathrm{Spf}\,\mathbb{Z}_p \times \mathrm{Spf}\,\mathbb{Z}_p = \mathrm{Spf}\,\underbrace{\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_p}$"

naively, it is $\mathbb{Z}_p$

b/c $p \otimes 1 = 1 \otimes p$ $(p = 1 + 1 + \cdots + 1)$.

$\rightsquigarrow$ Need Witt vectors.

For $R/\mathbb{F}_q$ perfect, we define

$$\text{"}R \otimes \mathbb{Z}_p\text{"} := W(R) = \sum_{n \geq 0} [a_n] p^n, \quad a_n \in R.$$

This is reasonable b/c $\mathbb{Z}_p = W(\mathbb{F}_p) \longrightarrow W(R)$

& $R \to W(R)$ morph of monoids.

Let $C/\mathbb{Q}_p$ be complete & alg closed.

$\rightsquigarrow$ $\mathcal{O}_C$ = ring of integers $\supseteq \mathfrak{m}_C$ = max ideal

$\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\mathfrak{m}_C = k$ res field.

Define the tilt $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p = \{(x_0, x_1, \dots) : x_i \in \mathcal{O}_C/p,\ x_i^p = x_{i+1}\}$

perfect ring of char p.

$\rightsquigarrow$ $C^\flat := \mathrm{Frac}\, \mathcal{O}_{C^\flat}$.

- $C^\flat$ is a complete alg closed valued field
  $p^\flat := (p, p^{1/p}, p^{1/p^2}, \dots)$, $|p^\flat| = |p|$.

In fact, $C^\flat \simeq \varprojlim_{x \mapsto x^p} C$, $\mathcal{O}_{C^\flat} \simeq \varprojlim_{x\mapsto x^p} \mathcal{O}_C \xrightarrow{\sim} \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$

Now define "$\mathbb{Z}_p \otimes \mathcal{O}_C$" $= W(\mathcal{O}_{C^\flat}) =: A_{\inf}$ à la Fontaine.
with its generators $(p, [p^\flat])$ (2-dim'l local ring).

Adic topology Features : $\cdot$ $\phi : A_{\inf} \longrightarrow A_{\inf}$

$\cdot$ $\theta : A_{\inf} \longrightarrow \mathcal{O}_C$

$$\sum_{n\geq 0} [a_n] p^n \longmapsto \sum_{n \geq 0} a_n^\# p^n$$

$\mathcal{O}_{C^\flat} \simeq \varprojlim_{x\mapsto x^p} \mathcal{O}_C \longrightarrow \mathcal{O}_C$

$x = (x_0, x_1, \dots) \longmapsto x_0 = x^\#$

$p^\flat \longmapsto p = p^{\flat\#}$

$\rightsquigarrow$ θ is surjective & $\ker\theta = (\xi)$ principal,

$$\xi = p - [p^\flat].$$

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat} \Rightarrow \varepsilon^\# = 1$. ζ_{p^n} = primitive p^nth root of 1.

$[\varepsilon] - 1 \in \ker\theta$, $(\varepsilon^{1/p})^\# = \zeta_p$.

$\rightsquigarrow$ another choice of ξ :

can take $\xi' = ([\varepsilon]-1)/([\varepsilon^{1/p}]-1)$.

The "graph" of $\mathrm{Spf}\, \mathcal{O}_C \longrightarrow \mathrm{Spf}\, \mathbb{Z}_p$ $(S \xrightarrow{x} X,\ \Gamma_x \in X \times S)$

$x_C \in$ "$\mathrm{Spf}\, \mathcal{O}_C \hat{\otimes} \mathbb{Z}_p$" = "$\mathrm{Spf}\, A_{\inf}$" where $x_C = (\xi) = \ker\theta$.

Can now define a "$\mathbb{Z}_p$-shtuka over $\mathcal{O}_{C^\flat}$ with a leg at $\mathrm{Spf}\, \mathcal{O}_{C^\flat} \longrightarrow \mathrm{Spf}\, \mathbb{Z}_p$".

represent $\mathbb{Z}_p \hookrightarrow \mathcal{O}_C$.

Def'n A Breuil-Kisin-Fargues mod (M, ϕ_M) is

- a free finite rank A_{inf}-mod M, with
- an isom $\phi_M : \phi^* M[\frac{1}{\phi(\xi)}] \longrightarrow M[\frac{1}{\phi(\xi)}]$ $\quad (\phi : A_{inf} \to A_{inf})$

  i.e. a ϕ-linear isom $M[\frac{1}{\xi}] \to M[\frac{1}{\phi(\xi)}]$.

"Realizations"

$$\begin{tikzcd} & & W(k)[\frac{1}{p}] = K \arrow[loop above, "\sigma"] \\ W(\mathcal{O}_C) = A_{inf} \arrow[urr, bend left, "x_k"] \arrow[rr, "x_C"] \arrow[drr, bend right, "x_{C^\flat}"'] & & \mathcal{O}_C \\ & & C^\flat \end{tikzcd}$$

crystalline realization

étale realization

- Crystalline: $(N, \phi_N) := (M, \phi_M) \otimes_{A_{inf}} K$

  where N a K-v.s. & $\phi_N : \sigma^* N \xrightarrow{\sim} N$ "isocrystal".

- Étale: $T := [M \otimes W(C^\flat)]^{\phi_M}$, $\mathrm{rk}_{\mathbb{Z}_p} T = \mathrm{rk}_{A_{inf}} M$.

§3 The Curve

Goal Recast the def'n of $\mathbb{Z}_p$-shtukas in terms of "$\mathrm{Spf}\, A_{inf} / \phi^{\mathbb{Z}}$".

Problems $x_{C^\flat}, x_k$ are fixed pts, so pass to $A_{inf}[\frac{1}{p[p^\flat]}]$.

$\phi^{\mathbb{Z}}$-orbit of x_C is dense (in Zariski topology).

$\nexists f \neq 0$, s.t. f has a zero at $\phi^n(x_C)$ for all $n \in \mathbb{Z}$.

Example $[\varepsilon] - 1$ has a zero at $x_C, \phi(x_C), \phi^2(x_C), \dots$

but NOT at $\phi^{-1}(x_C), \phi^{-2}(x_C), \dots$ $\quad (\theta([\varepsilon^{1/p^n}]) = \zeta_{p^n})$.

Want to define $t = \log[\varepsilon] = \lim_{n \to \infty} \frac{[\varepsilon^{p^n}] - 1}{p^n}$.

but this fails to converge in $A_{inf}[\frac{1}{p[p^\flat]}]$.

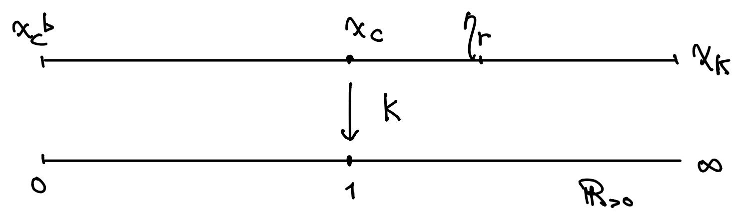
(so good top is the adic top.)

Lesson: Use $\mathrm{Spa}\, A_{\inf} = \{\text{cts valuations on } A_{\inf}\}$

- Points of $\mathrm{Spa}\, A_{\inf}$:
  - (1) x_k: unique non-analytic pt
  - (2) $x_{C^\flat}$, (3) x_C, (4) x_K
  - (5) $\forall r \in (0,\infty)$, let $\eta_r \in \mathrm{Spa}\, A_{\inf}$.

$$\Big|\sum_{n\geq 0}[a_n]p^n\Big|_{\eta_r} = \sup_n \frac{|x_n|^r}{p^n}.$$

Picture of $\mathrm{Spa}\, A_{\inf} \setminus \{x_k\}$:



with $\kappa(\eta_r) = r$, $\kappa(|\cdot|) = \dfrac{\log|[p^\flat]|}{\log|p|}$.

Def'n $\mathcal{Y}_{FF} = \big(\mathrm{Spa}\,(A_{\inf} \setminus \{|p[p^\flat]| = 0\})\big) \setminus \{x_K, x_{C^\flat}, x_k\}$

$\mathcal{X}_{FF} = \mathcal{Y}_{FF}/\phi^{\mathbb{Z}}$ Fargues–Fontaine curve.

Next time: relate BKF modules to vector bundles on $\mathcal{X}_{FF}$.