Local shtukas and the Langlands program (2/2)

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§1 Vector bundles on X_{FF}.

<u>Goal</u> To define $Sh(GL_n, \mu) \longrightarrow \operatorname{Spec} \mathbb{Q}_p$.

Given S, should classify:

$$\text{"}S \xrightarrow{x} \operatorname{Spec} \mathbb{Q}_p\text{"} \rightsquigarrow \Gamma_x \subseteq \text{"}\operatorname{Spec} \mathbb{Q}_p \times S\text{"}$$

- $\mathcal{E}$ v.b. of rk n on "$\operatorname{Spec} \mathbb{Q}_p \times S$".
- $f : \operatorname{Fr}_S^* \mathcal{E} \dashrightarrow \mathcal{E}$ isom outside Γ_x.

<u>Recall</u> $Y_{FF} = \operatorname{Spa} W(\mathcal{O}_C^\flat) \setminus \{ |p[p^\flat]| = 0 \}$

where $C/\mathbb{Q}_p$ alg closed & complete.

$\rightsquigarrow X_{FF} = Y_{FF}/\phi^{\mathbb{Z}}$.

Define $B = H^0(Y_{FF}, \mathcal{O}_{Y_{FF}})$.

In fact, $\exists$ an isom $(\underbrace{1+\mathfrak{m}_{C^\flat}}_{\mathbb{Z}_p\text{-mod \& gp of multi}}, \cdot) \xrightarrow{\sim} B^{\phi = p}$ b/w $\mathbb{Q}_p$-v.s. (Banach-Colmez spaces).

(indeed a $\mathbb{Q}_p$-v.s.).

via $z \longmapsto \log[z]$, $\phi(\log[z]) = p \log[z]$.

<u>Isocrystal</u> (N, σ_N): N v.s. $/ K = W(k)[\frac{1}{p}]$, $\sigma_N : \sigma^* N \xrightarrow{\sim} N$.

Isocrystals classified by $b \in GL_n(K) / b \sim g^\sigma b g = \underbrace{B(GL_n)}_{\text{Called Kottwitz set}}$

b: the matrix of σ_N. $\sigma : W(k) \longrightarrow W(k)$.

$\exists$ a functor (isocrystals) $\longrightarrow$ (vector bundles on $\mathcal{X}_{FF}$).

$(N, \sigma_N) \longmapsto N \otimes_K \mathcal{O}_{\mathcal{Y}_{FF}}$ descended through $\sigma_N \otimes \phi$.

$(N = K, \sigma_N e = \frac{1}{p} e) \longmapsto \mathcal{O}_{\mathcal{Y}_{FF}}(1)$.

$H^0(\mathcal{O}(1)) = B^{\phi = p}$ $\quad \frac{1}{p}\phi(\alpha e) = \alpha e$.

Thm (Fargues-Fontaine)

Every $\mathcal{F}$ is isom to $\mathcal{E}(N, \sigma_N)$.

Caveat : But the functor is not an equiv.

("not injective".)

If N is basic, then

$$\mathrm{Aut}(N, \sigma_N) \xrightarrow{\sim} \mathrm{Aut}\,\mathcal{E}(N, \sigma_N).$$

(note: $\mathrm{Aut}\,\mathcal{O}_X^n = GL_n(\mathbb{Q}_p)$.)

Thm (Fargues) Equivalence of Cats :

(i) BKF modules (M, ϕ_M) : $M = A_{inf}$-mod

& $\phi_M : M[\frac{1}{\xi}] \xrightarrow{\sim} M[\frac{1}{\phi(\xi)}]$ ϕ-linear.

(ii) $(T, \mathcal{E}, \beta)$: $T = \mathbb{Z}_p$-mod, free of fin rk,

$\mathcal{E}$ = v.b. on $\mathcal{X}_{FF}$.

$\beta : T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{X}_{FF} \setminus \{x_C\}} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{X}_{FF} \setminus \{x_C\}}$ mero at x_C.

From (i) to (ii) : taking $T = [M \otimes W(C^\flat)]^{\phi_M}$.

Example $A_{inf}\{1\} = (M, \phi_M)$, $M = A_{inf}\, e_M$, $\phi_M(e_M) = \frac{1}{\phi(\xi)} e_M$

Tate twist

where the choice $\xi = ([\varepsilon] - 1)/([\varepsilon^{1/p}] - 1)$

$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \mathcal{O}_{C^\flat}$.

$T \overset{\text{étale}}{=} \mathbb{Z}_p([\varepsilon]-1)e_M$, $(N, \phi_N) = (Ke_N, \phi_N(e_N) = p^{-1}e_N)$.

$\xi \mapsto p$ in $A_{\inf} \to K$.

étale end $\qquad\qquad\qquad\qquad\qquad$ crystal end

$$\begin{tikzcd} x_{C^\flat} & x_C \arrow[l, "(1)"'] \arrow[r, "(2)"] & x_K \end{tikzcd}$$

<u>Steps</u> (1) The conclusion: $T \hookrightarrow M$ induces $\mathcal{O}_{Y_{FF}} \xrightarrow{([\varepsilon]-1)} M|_{Y_{FF}}$.

$[\varepsilon]-1$: has zeros at $x_C, \phi(x_C), \phi^2(x_C), \cdots$.

(2) $M|_{Y_{FF}} \longrightarrow N \otimes_K \mathcal{O}_{Y_{FF}}$ with zeros at $\phi^{-1}(x_C), \phi^{-2}(x_C), \cdots$.

$e_M \longmapsto \frac{t}{[\varepsilon]-1} \cdot e_N$.

(3) Combine (1) & (2):

$$\mathcal{O}_{Y_{FF}} \xrightarrow[\phi]{t} N \otimes_K \mathcal{O}_{Y_{FF}}$$

has zeros at all $\phi^n(x_C)$, $n \in \mathbb{Z}$.

Descend to X_{FF}: $\mathcal{O}_{X_{FF}} \xrightarrow{t} \mathcal{O}(1)$

with zeros at $x_C \in X_{FF}$.

§2 Perfectoid spaces

We've constructed "$\mathrm{Spa}\,\mathbb{Z}_p \times \mathrm{Spa}\,\mathcal{O}_{C^\flat}$" $=: \mathrm{Spa}\,W(\mathcal{O}_{C^\flat})$

but not yet "$\mathrm{Spa}\,\mathbb{Z}_p \times \mathrm{Spa}\,\mathbb{Z}_p$".

Need perfectoid spaces.

<u>Def'n</u> A top ring A is <u>perf'd</u> if

(1) A is Tate ($A_0 \subseteq A$ open, A_0 has ϖ-adic top, $\varpi \in A^\times$).

e.g. · $A = \mathbb{Q}_p \supseteq \mathbb{Z}_p = A_0$.

· $A = \mathbb{Q}_p\langle T\rangle \supseteq \mathbb{Z}_p\langle T\rangle = A_0$.

(2) A is uniform.

(3) $\exists\, \varpi^p | p$, $A^\circ/\varpi \xrightarrow{\sim} A^\circ/\varpi^p$ isom.

$x \longmapsto x^p$

Build category of perfectoid spaces from $\mathrm{Spa}(A, A^+)$.

Given A, its tilt $A^\flat = \varprojlim_{x \mapsto x^p} A$, $S \longmapsto S^\flat$. $\quad A^\flat / \mathbb{F}_p$

preserving analytic top, étale top, etc.

Let Perf = cat of perf spaces $/\mathbb{F}_p$,

and define $\mathrm{Spd}\, \mathbb{Z}_p : \mathrm{Perf} \longrightarrow \mathrm{Sets}$

diamond spectrum

$$\begin{array}{ccc} & S \longmapsto \{(S^\sharp, i) \mid i : S \xrightarrow{\sim} S^{\sharp\flat},\ S^\sharp/\mathbb{Q}_p\}. \\ \uparrow & \\ \mathrm{Spd}\, \mathbb{Q}_p & \text{subsheaf} \end{array}$$

Use "v-top on Perf" $\approx$ fpqc top on schemes.

- $\mathrm{Spd}\, \mathbb{Q}_p$ is a "diamond",

$$\mathrm{Spd}\, \mathbb{Q}_p \simeq (\mathrm{Spd}\, \mathbb{Q}_p^{\mathrm{cycl},\flat}) / \mathbb{Z}_p^\times.$$

- For $C/\mathbb{F}_p$ alg closed,

$$\{\mathrm{Spa}\, C \longrightarrow \mathrm{Spd}\, \mathbb{Q}_p\} \simeq \{\mathbb{Q}_p^{\mathrm{cycl},\flat} \longrightarrow C\} / \mathbb{Z}_p^\times.$$
$$\simeq ((1 + \mathfrak{m}_C, \cdot) \setminus \{1\}) / \mathbb{Z}_p^\times.$$

Now "$\mathrm{Spd}\, \mathbb{Z}_p \times \mathrm{Spd}\, \mathbb{Z}_p$" makes sense.

§3 The local shtuka spaces

The relative curve

Given $S = \mathrm{Spa}(R, R^+)$ affinoid perf'd $/\mathbb{F}_p$

Let $\quad Y_{FF,S} = \mathrm{Spa}\, W(R^+) \setminus \{|p[\varpi]| = 0\} \not\longrightarrow S$

ϕ $\quad$ not over S.

$\rightsquigarrow X_{FF,S} = Y_{FF,S} / \phi^{\mathbb{Z}}$.

$\exists$ bijection $\{S \longrightarrow \mathrm{Spd}\, \mathbb{Q}_p\} \xrightarrow{\sim} \left\{\begin{array}{l} \text{Cartier divisors} \\ D_{S^\sharp} \subseteq Y_{FF,S} \text{ deg } 1 \end{array}\right\}.$

via $W(R^+) \xrightarrow{\theta} R^{\#,+}$, $D_{S^\#} = \ker\theta$.

The product $\underbrace{\mathrm{Spd}\,\mathbb{Q}_p}_{\text{Perf}} \times \underset{\uparrow}{S}$ makes sense
rep'ble in Perf.

$\rightsquigarrow$ $\mathcal{Y}^{\diamond}_{FF,S} \simeq \mathrm{Spd}\,\mathbb{Q}_p \times S$

a map $S \longrightarrow \mathrm{Spd}\,\mathbb{Q}_p$ is an untilt $S^\#$

the "graph of x" is $D_{S^\#} \subseteq \mathcal{Y}_{FF,S}$.

* Everything is def'd for shtukas.

Def'n (Local shtukas) Fix n, $b \in GL_n(\breve{\mathbb{Q}}_p)$ isocrystal.

$\breve{\mathbb{Q}}_p = W(\bar{\mathbb{F}}_p)[\frac{1}{p}]$.

Let $\mathrm{Sht}(GL_n, b, \mu) \longrightarrow \mathrm{Spd}\,\breve{\mathbb{Q}}_p$

points over $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_p}$ classify

$b \rightsquigarrow \mathcal{E}_b \in \mathrm{Bun}_n(\mathcal{X}_{FF,S})$.

- $x : S \longrightarrow \mathrm{Spd}\,\breve{\mathbb{Q}}_p$, i.e. $S^\# / \breve{\mathbb{Q}}_p$.
- $f : \mathcal{O}^n_{\mathcal{X}_{FF,S}} \dashrightarrow \mathcal{E}_b$ isom away from $D_{S^\#}$.

  at $D_{S^\#}$, f mero & bounded by μ.

note

$$\begin{array}{ccc} \overset{\mathrm{Aut}\,\mathcal{E}_b}{\curvearrowright} & & \\ \mathrm{Sht}(GL_n, b, \mu)_\infty & \circlearrowleft & GL_n(\mathbb{Q}_p) \\ \downarrow & & \cup\! \mid \\ & & H \\ \mathrm{Sht}(GL_n, b, \mu)_H & & \end{array}$$

(recall: $GL_n(\mathbb{Q}_p) = \mathrm{Aut}\,\mathcal{O}^n_{\mathcal{X}}$.)