Cohomology sheaves of stacks of shtukas (1/2)

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§1 Review

$X/\mathbb{F}_q$ proj sm geom conn curve,

F = func field of X, $\mathbb{A}$, $\mathbb{O}$ rings of adelès & integral adelès.

$G/\mathbb{F}_q$ split connected reductive gp.

Assume G semisimple. Consider the case without level str.

Fix $\ell \neq p$. Let $\hat{G}$ = dual Langlands gp of $G/\bar{\mathbb{Q}}_\ell$.

$\rightsquigarrow$ the space of autom forms

$$\underset{\substack{\uparrow \\ \text{cpt supp}}}{\mathrm{Func}_c}(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \bar{\mathbb{Q}}_\ell) = \mathrm{Func}_c(\mathrm{Bun}_G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$$

It is a $\bar{\mathbb{Q}}_\ell$-v.s., possibly of dim $= \infty$.

It is equipped w/ Hecke alg action (see Yun's talk).

$$\mathcal{H}_G := \underset{\substack{\uparrow \\ \text{conti fcns}}}{C_c}(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \bar{\mathbb{Q}}_\ell).$$

· Def'n of cohom of shtukas (generalized $\mathrm{Func}_c(\mathrm{Bun}_G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell)$).

Let I = finite set $\rightsquigarrow$ the stack of shtukas

Inductive lim of alg stacks: $\mathrm{Sht}_{G,I}$ Bound: given by rep'n of dual gps.

$$\mathrm{Sht}_{G,I} \longrightarrow X^I$$

Let $\underset{\substack{\uparrow \\ \text{cat of fin-dim'l } \bar{\mathbb{Q}}_\ell\text{-lin rep'n of } \hat{G}^I.}}{W \in \mathrm{Rep}(\hat{G}^I)} \rightsquigarrow \mathrm{Sht}_{G,I,\boxed{W}} \longrightarrow X^I$. serves as $\lambda \in X_*(T)^+$.

We have the bounded stacks of shtukas associated to I & W.

Also, $\mathrm{Sht}_{G,I,W}$ — "$IC_{G,I,W}$" (intersection complex sheaf)

$$\begin{array}{ccl} \mathrm{Sht}_{G,I,W} & & \\ p \downarrow & \rightsquigarrow & \mathcal{H}_{I,W} := Rp_!(IC_{G,I,W}) \\ X^I & & \end{array}$$

Cohom complex

$\forall$ degree $j \in \mathbb{Z}$, $\exists$ cohom sheaf $\mathcal{H}^j_{I,W} := R^j p_!(IC_{G,I,W})$

$\mathcal{H}^j_{I,W}$ is an ind-constructible $\bar{\mathbb{Q}}_\ell$-sheaf over X^I.

- Harder-Narasimhan stratification

$$\mathrm{Sht}_{G,I,W} = \bigcup_{\mu \in \underbrace{X_*(T)^+}_{\text{dom cowts of } G}} \mathrm{Sht}^{\le\mu}_{G,I,W}.$$

each of $\mathrm{Sht}^{\le\mu}_{G,I,W}$ is an open substack of fin type.

& $\mathcal{H}^j_{I,W} = \varinjlim_{\mu} \boxed{\mathcal{H}^{j,\le\mu}_{I,W}}$

$:= R^j p_!(IC_{G,I,W}|_{\mathrm{Sht}^{\le\mu}_{G,I,W}})$

Constructible $\bar{\mathbb{Q}}_\ell$-sheaf.

If $\mu_1 \le \mu_2 \rightsquigarrow \mathrm{Sht}^{\le\mu_1}_{G,I,W} \subseteq \mathrm{Sht}^{\le\mu_2}_{G,I,W}$ open

$\rightsquigarrow \mathcal{H}^{\le\mu_1}_{G,I,W} \longrightarrow \mathcal{H}^{\le\mu_2}_{G,I,W}$.

Consider

$$\begin{array}{ccc} \mathrm{Sht}^{\le\mu}_{G,I,W} & & ((x_i)_{i\in I}, \mathcal{G} \dashrightarrow {}^\tau\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Bun}^{\le\mu}_G & & \mathcal{G} \end{array}$$

(E.g. $G = SL_2$, $\mathrm{Bun}_{SL_2}(\bar{\mathbb{F}}_q) = \{\mathcal{O}(n) \oplus \mathcal{O}(-n) : n \in \mathbb{Z}_{\ge 0}\}$

$X = \mathbb{P}^1$, $\mu = (m, -m)$, $m \in \mathbb{Z}_{\ge 0}$

$\mathrm{Bun}^{\le\mu}_{SL_2}(\bar{\mathbb{F}}_q) = \{\mathcal{O}(n) \oplus \mathcal{O}(-n) : 0 \le n \le m\}$.)

$X \longleftarrow \eta \longleftarrow \bar{\eta}$ ← geom generic $\rightsquigarrow X^I \leftarrow \eta_I = \mathrm{Spec}\, F_I \longrightarrow \bar{\eta}_I = \mathrm{Spec}\, \bar{F_I}$.

η: generic pt; $\bar{\eta}$ "$=$" $\mathrm{Spec}\, \bar{F}$. η_I: generic; $\bar{\eta}_I$: geom generic

where F_I = func field of X^I.

Caution: $F_I \ne F^I$.

Define $H^j_{I,W} := \mathcal{H}^j_{I,W}|_{\bar\eta_I}$ the cohom gp.

It is a $\bar{\mathbb{Q}}_\ell$-v.s., may have ∞-dim.

<u>Rmk</u> When $I = \phi$, W = triv rep.

$$\begin{array}{c} \mathrm{Sht}_{G,\phi,W} = \mathrm{Bun}_G(\mathbb{F}_q). \\ \downarrow \\ \mathrm{Spec}\,\mathbb{F}_q = X^\phi \end{array} \Rightarrow \mathcal{H}^0_{I,W} = \mathrm{Func}_c(\mathrm{Bun}_G(\mathbb{F}_q), \bar{\mathbb{Q}}_\ell).$$

On $H^j_{I,W}$, there are:

- an action of the Hecke alg
- an action of partial Frob morphisms: $X^I \xrightarrow{\mathrm{Frob}_{\{i\}}} X^I$

$$(x_j)_{j\in I} \longmapsto (x_j')_{j\in I}$$

$$x_i' = \mathrm{Frob}(x_i),\quad x_j' = x_j \quad (i \neq j).$$

- an action of $\mathrm{Weil}(\eta_I, \bar\eta_I) = \mathrm{Weil}(\bar F_I / F_I)$.

$$\left(\begin{array}{ccc} \mathrm{Gal} & \longrightarrow & \hat{\mathbb{Z}} \\ \cup & & \cup \\ \mathrm{Weil} & \longrightarrow & \mathbb{Z} \end{array}\right).$$

these two actions + finiteness condition
+ Drinfeld's lemma

$\Rightarrow$ an action of $\mathrm{Weil}(\bar F/F)^I$.

<u>§2 Finiteness of $H^j_{I,W}$</u>

Method 1: Eichler-Shimura relations $\leftarrow$ Today.

Method 2: Constant term morphisms

Hecke operator:

Let v be a place of X.

$\rightsquigarrow$ Local Hecke alg $\mathcal{H}_{G,v} := C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \bar{\mathbb{Q}}_\ell)$.

(standard ver.) $\qquad$ conti func

Let $V \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$. By the Satake isomorphism,

V corresponds to a function in $\mathcal{H}_{G,v}$

which we denote by $h_{V,v}$.

We have the Hecke operator

for some $\kappa \in X_*(T)^+$ big enough s.t. $\forall \mu \in X_*(T)^+$

$$T(h_{V,v}) : \mathcal{H}_{I,W}^{j,\le\mu}\big|_{(X-v)^I} \longrightarrow \mathcal{H}_{I,W}^{j,\le\mu+\kappa}\big|_{(X-v)^I}$$

(a special case of the $\underbrace{\text{excursion operator}}_{\text{next time.}}$.)