Cohomology sheaves of stacks of shtukas (2/2)

Cong Xue

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<u>Setups</u> $\delta_V : \mathbb{1} \longrightarrow V \otimes V^*$ $\qquad$ V^* - dual rep'n of $V \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$.

$$1 \longmapsto \sum_i e_i \otimes e_i^*$$

* Creation operator $C^{\#}_{\delta_V}$: $\qquad$ (induced by)

$$(\bar{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}^{j, \leq\mu}_{I,W} \simeq \mathcal{H}^{j,\leq\mu}_{\{*\}\cup I, \mathbb{1}\boxtimes W} \xrightarrow{\delta_V \boxtimes \mathrm{Id}_W} \mathcal{H}^{j,\leq\mu}_{\{*\}\cup I, (V\otimes V^*)\boxtimes W}\big|_{\Delta(v)\times X^I}$$

$v \times X^I$ $\qquad$ functoriality. $\qquad$ $\Delta : X \xrightarrow{\mathrm{diag}} X \times X$.

$$\xrightarrow[\text{(factoriasation str)}]{\text{fusion}} \mathcal{H}^{j,\leq\mu}_{\{1,2\}\cup I, (V\boxtimes V^*)\boxtimes W}\big|_{\Delta(v)\times X^I}.$$

* Annahilation operator $C^{\flat}_{ev_V}$ $\qquad$ ($ev_V : V \otimes V^* \longrightarrow \mathbb{1}$).

$$\begin{array}{ccc} \mathcal{H}^{j}_{\{*\}\cup I, (V\boxtimes V^*)\boxtimes W} & \simeq & \mathcal{H}^{j}_{\{1,2\}\cup I, (V\boxtimes V^*)\boxtimes W}\big|_{\Delta(v)\times X^I} \\ \Big\downarrow{\scriptstyle ev_V \boxtimes \mathrm{Id}_W} & & \Big\downarrow{\scriptstyle C^{\flat}_{ev_V}} \\ \mathcal{H}^{j}_{\{*\}\cup I, \mathbb{1}\boxtimes W} & \simeq & (\bar{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}^{j}_{I,W} \end{array}$$

* The excursion operator $S_{V,v}$ ass to V and v is the composition of morphisms of sheaves on $v \times X^I$:

$$(\bar{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}^{j}_{I,W} \xrightarrow{C^{\#}_{\delta_V}} \mathcal{H}^{j}_{\{1,2\}\cup I, V\boxtimes V^* \boxtimes W}\big|_{\Delta(v)\times X^I}$$

(changes the HN truncation) $\quad \xrightarrow{F^{\deg v}_{\{1\}}} \mathcal{H}^{j}_{\{1,2\}\cup I, V\boxtimes V^*\boxtimes W}\big|_{\Delta(v)\times X^I}$

partial Frob morph $\qquad$ $\mathrm{Frob}^{\deg(v)}(v) = v$.

$$\xrightarrow{C^{\flat}_{ev_V}} (\bar{\mathbb{Q}}_\ell)_v \boxtimes \mathcal{H}^{j}_{I,W}.$$

* $S_{V,v}$ descends to a morphism of sheaves on $\boxed{X^I}$:

$$S_{V,v} : \mathcal{H}^{j,\leq\mu}_{I,W} \longrightarrow \mathcal{H}^{j,\leq\mu+\kappa}_{I,W}$$

<u>Prop</u> (V. Lafforgue) The operator $S_{V,v}$ extends $T(h_{V,v})$ $\qquad$ morph of sheaves on $(X-v)^I$.

the Hecke operator def'd outside v.

Prop (V. Lafforgue) Let $W = \boxtimes_{i\in I} W_i$, with $W_i \in \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$.
Let $(v_i)_{i\in I}$ be a family of closed pts of X.
Then $\exists\, \kappa \in X_*(T)^+$ big enough s.t. $\forall \mu \in X_*(T)^+$, $\forall i \in I$, we have

$$\sum_{\alpha=0}^{\dim W_i} (-1)^\alpha S_{\Lambda^{\dim W_i-\alpha} W_i,\, v_i} \circ \left(F_{\{i\}}^{\deg(v_i)}\right)^\alpha = 0.$$

as polynomial in $\mathrm{Hom}\left(\mathcal{H}_{I,W}^{j,\le\mu}\big|_{\prod_{i\in I} v_i},\ \mathcal{H}_{I,W}^{j,\le\mu+\kappa}\big|_{\prod_{i\in I} v_i}\right)$

Rmk These two prop'ns together are called the E-S relation.

For any $\mu \in X_*(T)^+$, we choose a dense open subsch Ω of X^I s.t. $\mathcal{H}_{I,W}^{j,\le\mu}\big|_\Omega$ is smooth.

We chose a closed pt $v \in \Omega$. $\rightsquigarrow$

$$\begin{array}{ccc} X^I & \xrightarrow{pr_i} & X \\ v & \longmapsto & x_i \end{array}$$

Define $\quad \mathfrak{M}_\mu := \sum_{(n_i)\in\mathbb{N}^I} \left(\bigotimes_{i\in I} \mathcal{H}_{G,v_i}\right) \circ \left(\prod_{i\in I} F_{\{i\}}^{n_i}\right) \mathcal{H}_{I,W}^{j,\le\mu}\big|_{\bar\eta_I}$

(i) stable by partial Frob

(ii) By ES relations, $\mathfrak{M}_\mu$ is of finite type as $\bigotimes_{i\in I}\mathcal{H}_{G,v_i}$-mod.

Also, $\mathcal{H}_{I,W}^{j}\big|_{\bar\eta_I} = \varinjlim_\mu \mathfrak{M}_\mu$ by construction.

Punchline Smoothness $\Rightarrow$ can pass from generic fiber to special fiber.

§3 Drinfeld's lemma and the action of Weil gps

$\bar\eta_I \longrightarrow \eta_I \longrightarrow X^I$, $\quad \bar\eta \longrightarrow \eta \longrightarrow X$ $\rightsquigarrow$

generic pt $\qquad$ generic pt.

We have a commutative diagram

$$\begin{tikzcd} \bar\eta_I \arrow[r] \arrow[d] & \eta_I \arrow[r] \arrow[d] & X^I \arrow[d, "pr_i"] \\ \bar\eta \arrow[r] & \eta \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{Weil}(\eta_I, \bar\eta_I) \arrow[r] & \mathbb{Z} \arrow[ddr, bend left] & \\
 & & & \hat{\mathbb{Z}} \arrow[d, equal] & \\
0 \arrow[r] & \pi_1^{geom}(\eta_I, \bar\eta_I) \arrow[uur] \arrow[r] \arrow[d] & \pi_1(\eta_I, \bar\eta_I) \arrow[r] \arrow[d] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] \arrow[d, "\mathrm{diag}"] & 0 \\
0 \arrow[r] & \pi_1^{geom}(\eta, \bar\eta)^I \arrow[r] \arrow[d, equal] & \pi_1(\eta, \bar\eta)^I \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)^I \arrow[r] \arrow[d, equal] & 0 \\
 & \pi_1(\eta \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q, \bar\eta)^I \arrow[dr] & \mathrm{Gal}(\bar F/F)^I & \hat{\mathbb{Z}}^I & \\
 & & \mathrm{Weil}(\eta, \bar\eta)^I \arrow[r] & \mathbb{Z}^I \arrow[uur, bend right] &
\end{tikzcd}$$

Define partial Frob + Weil.

$$\begin{array}{ccl}
\delta & \in & \mathrm{FWeil}(\eta_I, \bar\eta_I) := \left\{ \delta \in \mathrm{Aut}_{\bar{\mathbb{F}}_q}(\bar F_I) \;\middle|\; \exists (n_i) \in \mathbb{Z}^I \text{ s.t. } \delta|_{(F_I)^{perfection}} = \prod_{i \in I} \mathrm{Frob}_{\{i\}}^{-n_i} \right\} \\
\downarrow & & \downarrow \quad (\text{fix } sp : \bar\eta_I \to \Delta(\bar\eta) \text{ the specialization}) \\
(\mathrm{Frob}_{\{i\}}^{n_i} \circ \delta|_{\bar F_i}) & \in & \mathrm{Weil}(\eta, \bar\eta)
\end{array}$$

where $\Delta : X \hookrightarrow X^I$, $\bar F \otimes \cdots \otimes \bar F \subset \bar F_I$.

Let $\mathcal{G}$ be a $\bar{\mathbb{Q}}_\ell$-sheaf over η_I, equipped w/ an action of partial Frob.
i.e. $F_{\{i\}} : (\mathrm{Frob}_{\{i\}}^*)\, \mathcal{G} \xrightarrow{\sim} \mathcal{G}$ & $\mathrm{Frob}_{\{i\}} : X^I \longrightarrow X^I$
Then $\mathcal{G}|_{\bar\eta_I}$ is equipped with an action of $\mathrm{FWeil}(\eta_I, \bar\eta_I)$.
on generic fibers

Lem 1 If a finite type $\bar{\mathbb{Z}}_\ell$-mod is equipped with a conti $\mathrm{FWeil}(\eta_I, \bar\eta_I)$-action, then it is equipped w/ an action of $\mathrm{Gal}(\bar F/F)^I$.

Lem 2 If a fin-dim'l $\bar{\mathbb{Q}}_\ell$-v.s. is equipped with a conti $\mathrm{FWeil}(\eta_I, \bar\eta_I)$-action, then it is equipped w/ an action of $\mathrm{Weil}(\bar F/F)^I$.

Lem 3 If a finite type module over a f.g. commutative $\bar{\mathbb{Q}}_\ell$-alg is equipped with a conti $\mathrm{FWeil}(\eta_I, \bar{\eta}_I)$-action, (e.g. local Hecke alg.) then it is equipped w/ an action of $\mathrm{Weil}(\bar{F}/F)^I$.

$$\rightsquigarrow \mathcal{H}^j_{I,W}|_{\eta_I} = \varinjlim_{\mu} m_\mu \overset{\text{Lem 3}}{\Longrightarrow} \mathcal{H}^j_{I,W}|_{\eta_I} \text{ is equipped with a } \mathrm{Weil}(\bar{F}/F)^I\text{-action.}$$

apply to each m_μ

Application 1

Thm The ind-constructible $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{H}^j_{I,W}$ over X^I is ind-lisse.

inductive lim of lisse sheaves.

Application 2

Can extend V. Lafforgue excursion operator on $\mathcal{H}^j_{I,W}|_{\bar{\eta}_I}$.

$$\begin{tikzcd} \mathcal{H}_{I,V} \arrow[r, "\text{creation}"] & \mathcal{H}_{J\cup I, V\boxtimes W} \arrow[r, "(\gamma_i)"] & \mathcal{H}_{J\cup I, V\boxtimes W} \arrow[d, "\text{annihilation}"] \\ & & \mathcal{H}_{J,V}. \end{tikzcd}$$