Introduction to shtukas and their moduli (1/3)

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<u>Setups</u> Let $X/k = \mathbb{F}_q$ complete, geom conn curve

$F = k(X) \rightsquigarrow F_x, \mathcal{O}_x, k_x$.

Let G = split reductive gp $/k$.

$C([G]) \circlearrowleft G(\mathbb{A})$.

Weil's dictionary for split red gp:

$G(F)\backslash(G(\mathbb{A})/K) \xleftrightarrow{\sim} \{G\text{-bundles on } X\}$ equiv of groupoids

$K = \prod_{x \in |X|} G(\mathcal{O}_x)$

$G(F)$ acts on $G(\mathbb{A})/K$ (as a set).

Consider Bun_G = moduli stack of G-bundles on X.

<u>Def'n of Bun_G</u> (need functor of pts)

$\mathrm{Bun}_G(S) = \{\mathcal{E} \to X \times S \ \ G\text{-torsor}\}$ (S can be $\mathrm{Spec}\, k$)

<u>Fact</u> Bun_G is an Artin (or alg) stack

loc. of fin type, smooth $/\mathrm{Spec}\, k$.

(When $G = GL_n$:

Bun_{GL_n} = moduli of vec bundles of rk n.)

Weil's dictionary is given by:

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(\mathcal{O}^n, \mathcal{V}) & \longleftarrow & \mathcal{V} \\ & \searrow & \downarrow \\ & & X \end{array} \qquad \begin{array}{ccc} \mathcal{E} & \longmapsto & \mathcal{V} = \mathcal{E} \overset{GL_n}{\times} V_{std} \\ \downarrow & \swarrow & \\ X & & \end{array}$$

(quotient the diag action by GL_n.)

E.g. $\mathrm{Bun}_{Sp_{2n}}(S) = \left\{ \begin{array}{l} \mathcal{V} \xrightarrow{\text{rk } 2n} X \times S \text{ via perfect pairing} \\ \mathcal{V} \otimes_{\mathcal{O}_{X\times S}} \mathcal{V} \longrightarrow \mathcal{O}_{X\times S} \text{ alternating perfect pairing} \end{array} \right\}$

Plan
- Shtukas with 1 leg & no bound
- Shtukas with 1 leg & bound
- Shtukas with more legs

§ Hecke stacks for G-bundles

$$\mathrm{Hk}_G(S) = \{ (x, \mathcal{E}, \mathcal{E}', \alpha : \mathcal{E}|_{X\times S \setminus \Gamma(x)} \xrightarrow{\sim} \mathcal{E}'|_{X\times S\setminus\Gamma(x)}) \}$$

($\mathcal{E}, \mathcal{E}' \downarrow X \times S$)

$$\begin{array}{ccccc} & & X & \leftarrow & \mathrm{Hk}_G \\ & & & p \swarrow & & \searrow p' \\ & & \mathrm{Bun}_G & & & & \mathrm{Bun}_G \end{array}$$

Here $S \xrightarrow{x} X \rightsquigarrow \Gamma(x) \hookrightarrow S \times X$ graph of x.

Def'n Sht_G with one leg is def'd by the Cartesian diag

$$\begin{array}{ccc} \mathrm{Sht}_G & \longrightarrow & \mathrm{Hk}_G \\ \downarrow & \ulcorner & \downarrow (p, p') \\ \mathrm{Bun}_G & \xrightarrow{(id, Fr)} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array}$$

Any $S/k = \mathbb{F}_q \rightsquigarrow Fr_S : S \longrightarrow S \quad \longleftrightarrow \quad \mathcal{O}_S \longleftarrow \mathcal{O}_S$, $\ f^q \longleftarrow f$

Calculation $\mathrm{Sht}_G(S) = \{ (x, \mathcal{E}, \mathcal{E}', \alpha : \mathcal{E}|_{X\setminus x} \xrightarrow{\sim} \mathcal{E}'|_{X \setminus x} \}$

$\iota : Fr_{\mathrm{Bun}_G}(\mathcal{E}) \cong \mathcal{E}'$.

Equiv'ly: $(x, \mathcal{E} \overset{\alpha}{\underset{X\setminus x}{\dashrightarrow}} {}^{\tau}\mathcal{E} = (id \times Fr)^* \mathcal{E})$ ← no need to introduce $\mathcal{E}'$.

with $X \times S \xrightarrow{id \times Fr} X \times S$

We introduce bounds for geom interpretation of shtukas.

Affine Grassmanian $G = GL_n$, $\mathrm{Gr}_n(k) = \{ \mathcal{O}_x\text{-lattices } \Lambda \subset F_x^{\oplus n} \text{ of full rank} \}$

(Gr_n ↑ $\mathrm{Gr}_{GL_n}(k)$)

Then $\boxed{Gr_G(k) = G(F_x)/G(\mathcal{O}_x)} \ni g \longleftrightarrow \Lambda = g\cdot \mathcal{O}_x^{\oplus n}$.

Consider $\lambda = (d_1 \geq d_2 \geq \cdots \geq d_n)$, $d_i \in \mathbb{Z}$

If $d_1 \geq \cdots \geq d_n \geq 0$, then, with $\mathcal{O}_x = k_x[\![t]\!]$, — implies $t^{d_1}\mathcal{O}_x^{\oplus n} \subseteq \Lambda \subseteq \mathcal{O}_x^{\oplus n}$.

$$Gr_{n,\lambda}(k) = \left\{ \Lambda \subseteq \mathcal{O}_x^{\oplus n} : \begin{array}{l} \boxed{\mathcal{O}_x^{\oplus n}/\Lambda} \circlearrowleft t \text{ nilp operator} \\ \text{with Jordan blocks of sizes } d_1, \cdots, d_n \end{array} \right\}$$

($Gr_{n,\lambda}(k)$ ← $Gr_{GL_n,\lambda}(k)$)

Also, $\lambda + d = (d_1+d, \cdots, d_n+d) \geq 0$

$\rightsquigarrow Gr_{n,\lambda+d} = \{ t^{-d}\Lambda : \Lambda \in Gr_{n,\lambda} \}$

We get a decomp:

$$Gr_n(k) = \coprod_{\lambda = (d_1 \geq \cdots \geq d_n)} Gr_{n,\lambda}(k).$$

For general G: $Gr_G = G(\!(t)\!)/G[\![t]\!]$

(where $G(\!(t)\!)(k) = G(k(\!(t)\!))$ affine Kac-Moody Lie alg)

↑ adding double brackets: "fin-dim $\rightsquigarrow$ aff"

doesn't mean the var is affine

Fact $Gr_{n,\lambda}(k)$ is a subvar in usual $Gr(n)$ for proj varieties.

We have $Gr_G = \coprod_{\lambda \in X_*(T)^+} \boxed{Gr_{G,\lambda}}$, T = max'l torus

($Gr_{G,\lambda}$ ↑ sheafification to get left $G[\![t]\!]$-orbs)

$X_*(T)^+ = \{\boxed{\text{dominant}} \text{ cochar lattices}\}$ ← $\langle \lambda, \alpha_i \rangle \geq 0$

$\cap$

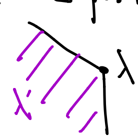
$X_*(T) = \{\text{cochar lattices}\}$.

$\rightsquigarrow$ a stratification of Gr_G (not a disj union of vars)

Can talk about $Gr_{G,\leq\lambda} = \overline{Gr_{G,\lambda}} = \bigcup_{\lambda' \leq \lambda} Gr_{G,\lambda'}$

($Gr_{G,\leq\lambda}$ ↑ proj var, dim $\langle 2\rho, \lambda \rangle$)

$\lambda' \leq \lambda$: $\lambda' - \lambda = \sum$ positive coroots



Fact $\dim Hk_{X,\leq\lambda} = \dim Bun_G + \dim Gr_{G,\leq\lambda}$

$= \dim G \cdot (\text{genus}(X) - 1) + \langle 2\rho, \lambda \rangle$

$\mathrm{Hk}_{\le\lambda}$: $\mathrm{Hk} \dashrightarrow \mathrm{Gr}$

$$\mathrm{Hk}_x = \{(\mathcal{E} \overset{\alpha}{\underset{X\setminus x}{\dashrightarrow}} \mathcal{E}')\} \xrightarrow{ev_x} \{(\mathcal{E}|_{D_x} \overset{\alpha}{\underset{D_x^\times}{\dashrightarrow}} \mathcal{E}'|_{D_x})\} \simeq \mathrm{QGr}_x$$

where $D_x^\times = \mathrm{Spec}\, F_x$, $D_x = \mathrm{Spec}\, \mathcal{O}_x$.

$D_x^\times$: punctured; D_x: formal disc

$\mathrm{QGr}_x = \underbrace{G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!]}$ with an unknown dim,

$\mathrm{QGr}(R) = (G(R[\![t]\!])\backslash G(R(\!(t)\!))/G(R[\![t]\!]))^{\text{sheafification}}$

For $G[\![t]\!] \curvearrowright \mathrm{Gr}_G$, orbits $\longleftrightarrow X_*(T)^+$.

Can consider $\mathrm{Hk}_{x,\le\lambda} = ev_x^{-1}(\mathrm{QGr}_{x,\le\lambda})$

Let x move on X.

$$\mathrm{Hk} \xrightarrow{ev} \mathrm{QGr}/\mathrm{Aut}(D) \qquad (D = \text{formal disc}).$$

(not an essential description; should look at $\mathrm{Hk}_{\le\lambda}$).

Example $G = GL_n$, $\lambda = (1, 0, \cdots, 0)$

$$\mathrm{Hk}_{\le\lambda} = \{x,\ V \overset{\alpha}{\hookleftarrow} V' : \text{coker } \alpha \text{ skyscraper, supp'd on } \Gamma(x), \text{ of rk } 1\}$$

Def

$$\begin{array}{ccc} \mathrm{Sht}_G^{\le\lambda} & \longrightarrow & \mathrm{Hk}^{\le\lambda} \\ \downarrow & \ulcorner & \downarrow (p, p') \\ \mathrm{Bun}_G & \xrightarrow{(id, Fr)} & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{array}$$

Resume on : $G = GL_n$, $\lambda = (1, 0, \cdots, 0)$

$$\Rightarrow \mathrm{Sht}_G^{\le\lambda} = \{x,\ V \hookleftarrow {}^\tau V \text{ s.t. } \deg {}^\tau V = \deg V - 1\}$$

impossible! b/c $\deg {}^\tau V = \deg V$.

τ: NOT the Frob on X, but the Frob on test scheme

$$\Rightarrow \mathrm{Sht}_G^{\le\lambda} = \phi.$$

Two ways to remedy : ① $\lambda = (1, 0, \cdots, 0, -1)$ $\rightsquigarrow$ to make $\deg {}^\tau V = \deg V$.

② iterated version.

Rmk. $\lambda = (1, 0, \dots, 0, -1)$

$$\rightsquigarrow \quad \mathcal{E} \overset{\le \lambda}{\dashrightarrow} \mathcal{E}' \quad \text{means} \quad \mathcal{E} \cap \mathcal{E}' \underset{\textcircled{\le 1}}{\subseteq} \mathcal{E}, \quad \mathcal{E} \cap \mathcal{E}' \underset{\textcircled{\le 1}}{\subseteq} \mathcal{E}'$$

Hecke iterated ver

$$\mathrm{Hk}^{(1,\dots,r)} = \left\{ \begin{array}{l} x_1, \dots, x_r, \text{ together with} \\ \mathcal{E}_0 \underset{X \setminus x_1}{\overset{\alpha_1}{\dashrightarrow}} \mathcal{E}_1 \underset{X \setminus x_2}{\overset{\alpha_2}{\dashrightarrow}} \mathcal{E}_2 \dashrightarrow \cdots \underset{X \setminus x_r}{\overset{\alpha_r}{\dashrightarrow}} \mathcal{E}_r \end{array} \right\}$$

bounded by $\lambda = (\lambda_1, \dots, \lambda_r)$, $\lambda_i \in X_*(T)^{dom}$

$$\mathrm{Hk}^{(1,\dots,r), \le \lambda} \rightsquigarrow \mathrm{Sht}_G^{(1,\dots,r), \le \lambda}.$$

E.g. $G = GL_1$, $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$.

$$\mathrm{Sht}_{GL_1}^{(1,\dots,r),\lambda} = \{ x_1, \dots, x_r,\ \mathcal{L}_0 \overset{\alpha_1}{\dashrightarrow} \mathcal{L}_1 \overset{\alpha_2}{\dashrightarrow} \cdots \overset{\alpha_r}{\dashrightarrow} \mathcal{L}_r \overset{?}{\cong} {}^\tau\mathcal{L}_0 \text{ s.t. } -\mathrm{div}(\alpha_i) = \lambda_i x_i \}$$

on GL_1, no way to compare dom coroots

$$\rightsquigarrow \mathcal{L}_0 \overset{\alpha_1}{\dashrightarrow} \mathcal{L}_0(-\lambda_1 x_1) \overset{\alpha_2}{\dashrightarrow} \cdots \overset{\alpha_r}{\dashrightarrow} \mathcal{L}_r = \mathcal{L}_0\left(-\sum \lambda_i x_i\right) \overset{?}{\cong} {}^\tau\mathcal{L}_0.$$

$\rightsquigarrow$ get the equation ${}^\tau\mathcal{L}_0 \otimes \mathcal{L}_0^{-1} = \mathcal{O}(-\sum \lambda_i x_i)$.

Consider

$$\begin{array}{ccccc} \mathrm{Sht}_{GL_1}^{(1,\dots,r),\lambda} & \longrightarrow & \mathrm{Pic}(X) & & \mathcal{L}_0 \\ \downarrow & \Gamma & \downarrow \mathrm{Lang} & & \downarrow \\ X^r & \longrightarrow & \mathrm{Pic}(X) & & {}^\tau\mathcal{L}_0 \otimes \mathcal{L}_0^{-1} \quad (\text{deg zero}) \\ (x_i) & \longmapsto & \mathcal{O}(-\sum \lambda_i x_i). & & \text{b/c } \deg {}^\tau\mathcal{L}_0 = \deg \mathcal{L}_0 \end{array}$$

Cor (1) Since Lang map lands in $\mathrm{Pic}^0(X)$,

$$\sum \lambda_i \neq 0 \Rightarrow \mathrm{Sht}_{GL_1}^{(1,\dots,r),\lambda} = \emptyset$$

(b/c we didn't choose a correct λ).

(2) $\sum \lambda_i = 0$ (can take $r=0$) $\Rightarrow \mathrm{Sht}_{GL_1}^{(1,\dots,r),\lambda} \neq \emptyset$.

(Fact Lang isog is a univ abelian cover of Pic.)

$$\rightsquigarrow \mathrm{Sht}_{GL_1}^{(1,\dots,r),\lambda} \longrightarrow X^r \quad (x_i = \text{leg}).$$

is torsor for $\mathrm{Pic}(X)(\mathbb{F}_q)$.