# Introduction to shtukas and their moduli (2/3)

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July 14

## §1 Geometric constructions

- Drinfeld's ver of shtukas (more concrete)

$G = GL_n$, 2 legs, $\lambda_1 = (1, 0, \cdots, 0)$, $\lambda_2 = (0, \cdots, 0, -1)$, $\underline{\lambda} = (\lambda_1, \lambda_2)$.

fundamental dom cowts.

$\rightsquigarrow$ rep of $\hat{G}$ corresponding to Std & $\mathrm{Std}^*$.

Consider $\mathrm{Sht}_{GL_n}^{(1,2),\underline{\lambda}} = \left\{ x_1, x_2, \text{ with } \boxed{\mathcal{E}_0 \xhookleftarrow{\alpha_1} \mathcal{E}_1 \xhookrightarrow{\alpha_2} \mathcal{E}_2 \overset{2}{\cong} {}^{\tau}\mathcal{E}_0} \right\}$ $\left( \text{so } \begin{matrix} \mathrm{Sht}_{GL_n}^{(1,2),\underline{\lambda}} \\ \downarrow \text{ rel dim } 2(n-1) \\ X^2 \end{matrix} \right)$.

Conditions : $\mathrm{coker}(\alpha_i)$ = skyscraper of length 1 at $x_i$.

" 2 bundles & sth smaller"

- Relation with Drinfeld modules :

$S = \mathrm{Sch}/k$. Choose $\infty \in X(k)$, $A = \Gamma(X \backslash \infty, \mathcal{O})$.

breaking the symmetry.

**Def'n** A Drinfeld A-mod /S is a pair $(\mathcal{G}, 2)$

where $\mathcal{G}$ gp sch, loc. $\cong \mathbb{G}_a$

$2 : A \longrightarrow \mathrm{End}_S(\mathcal{G})$.

$A \circlearrowleft \mathrm{Lie}(\mathcal{G}/S)$

$A \to \mathcal{O}_S$, $S \xrightarrow{x} X \backslash \infty$

Say $(\mathcal{G}, 2)$ has rank ⓝ if $\forall s = \mathrm{Spec}\, k \in S$ (geom pt).

$$\begin{tikzcd} A \arrow[r] & \mathrm{End}(\mathbb{G}_{a,s}) = k\langle\tau\rangle \\ a \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \text{poly in } \tau \text{ of deg} = ⓝ \cdot \mathrm{ord}_\infty(a). \end{tikzcd}$$

on which $\tau \cdot a = a^q \cdot \tau$, $q = \#k$.

We are to propose an equiv of grpoids :

$$\mathrm{DrMod}_n(S) \longleftrightarrow \mathrm{Sht}_n^{Dr}(S).$$

Define

$$\mathrm{Sht}_2^{\mathrm{Dr}}(S) = \left\{ \begin{array}{c} x \neq \infty \quad \text{(think } x_2 = \infty) \\ \begin{tikzcd} & & & & \mathcal{E}_0 \otimes \mathcal{O}(\{\infty\}) \arrow[d, equal] & \\ \cdots \arrow[r, phantom, "\subset"] & \mathcal{E}_0(-\frac{1}{2}) \arrow[r, phantom, "\subset_\infty"] & \boxed{\mathcal{E}_0} \arrow[r, phantom, "\subset_\infty"] & \mathcal{E}_0(\frac{1}{2}) \arrow[r, phantom, "\subset_\infty"] & \mathcal{E}_0(1) \arrow[r, phantom, "\subset"] & \cdots \\ \cdots \arrow[r, phantom, "\subset"] \arrow[ur] & {}^\tau\mathcal{E}_0(-\frac{1}{2}) \arrow[r, phantom, "\subset"] \arrow[ur] & {}^\tau\mathcal{E}_0 \arrow[r, phantom, "\subset"] \arrow[ur] & {}^\tau\mathcal{E}_0(\frac{1}{2}) \arrow[r, phantom, "\subset"] \arrow[ur] & {}^\tau\mathcal{E}_0(1) \arrow[r, phantom, "\subset"] \arrow[ur] & \cdots \end{tikzcd} \\ \text{“ 2 bundles \& sth smaller”} \\ \text{s.t. all cokers = skyscrapers at } x \end{array} \right\}$$

(all the differences are at $\infty$)

Have

$$\mathrm{Sht}_2^{\mathrm{Dr}} \xhookrightarrow[\text{embedding}]{\text{closed}} \mathrm{Sht}_{GL_2}^{(1,2),\delta}\Big|_{(X\setminus\infty)\times\{\infty\}}$$

$\dim = 2$ $\qquad\qquad$ $\dim = 3$.

A hint to proceed on:

$$\begin{tikzcd} \mathrm{DrMod}_n(S) \arrow[dr, hook] & & \mathrm{Sht}_n^{\mathrm{Dr}}(S) \arrow[dl, hook'] \\ & M \arrow[loop left, "A"] \arrow[loop right, "\mathcal{O}_S\{\tau\}"] & \end{tikzcd}$$

$M = \Gamma(X\setminus\infty, \mathcal{E}_0)$ as an $A$-mod.

$\mathcal{O}_S\{\tau\}$ (loc. free of rk 1)

$\parallel$

$\underline{\mathrm{End}}_S(\mathbb{G}_a)$ as a sheaf.

· Isoshtukas (coarse description of $\mathrm{Sht}_G(\bar{k})$)

$G = GL_n$.

$\breve{F}$-v.s., $n$-dim'l

$$\left( \begin{tikzcd} \mathcal{E} \arrow[rr, dashed] \arrow[dr, no head] & & {}^\tau\mathcal{E} = (\mathrm{id}_X \times \mathrm{Fr}_{\bar{k}/k})^* \mathcal{E} \arrow[dl, no head] \\ & X_{\bar{k}} & \end{tikzcd} \right) \xrightarrow[\text{generic pt of } X_{\bar{k}}]{\text{restr. to}} \left( \begin{array}{l} V \xleftarrow[\sim]{\phi} V : F\text{-linear} \\ \& \ \mathrm{Fr}_{\bar{k}/k}\text{-semilinear} \end{array} \right)$$

has func field $F \otimes_k \bar{k} = \breve{F}$.

$= \mathrm{IsoSht}(F)$.

For any $x \in X(k)$, $\quad \mathrm{IsoSht}(F) \xrightarrow{\ \omega_x\ } \mathrm{Isoc}(F_x)$

$$(V, \phi) \longmapsto (V \otimes_{\breve{F}} \breve{F}_x, \ \phi \otimes \mathrm{Fr})$$

If $x$ not rational, need to raise this to a power

Dieudonné - Manin classification:

$$\begin{tikzcd} \text{simple } \mathrm{Isoc}(F_x) \arrow[r, leftrightarrow] & \mathbb{Q} \ \text{(slopes)} \\ M(\lambda) & \lambda \arrow[l, mapsto] \end{tikzcd}$$

$\dim_{\breve{F}_x} M(\lambda) =$ denom of $\lambda$, $\mathrm{End}(M(\lambda))$ central div alg $/ F_x$

invariant $= -\lambda \pmod{\mathbb{Z}}$.

### · Drinfeld's description of IsoSht(F)

- IsoSht(F) is semisimple
- simple obj $\longleftrightarrow (\bar{F}^{sep,\times} \otimes_{\mathbb{Z}} \mathbb{Q}) / \mathrm{Gal}(\bar{F}/F)$.

$$(V, \phi) \longmapsto a \in \boxed{L_a}^{\times} \otimes \mathbb{Q}, \quad L_a \text{ smallest.}$$

has CM by $L_a$.  determined by a (up to Gal)

$\mathrm{End}(V,\phi)$, central div alg $/L_a$

loc inv at $y \in |L_a|$ is $-\mathrm{ord}_y(a) \cdot [k_y : k] \in \mathbb{Q}/\mathbb{Z}$.

$\omega_y(V,\phi) \in \mathrm{Isoc}(L_{a,y})$ $\underbrace{\text{isoclinic}}_{}$, slope $= -\mathrm{ord}_y(a) \cdot [k_y : k]$.

= direct sum of the same simple objs.

### · Geometric properties of Sht (Varshavsky)

<u>Fact</u> $\mathrm{Sht}_G^{(1,\dots,r),\le\lambda}$ is an alg stack, locally of fin type $/X^r$ with

$$\text{rel. dim} = \begin{cases} -\infty, \ (\mathrm{Sht} = \phi) & \text{if } \sum \lambda_i \notin \text{Coroot lattice} \\ \sum_{i=1}^{r} \langle 2\rho, \lambda_i \rangle, & \text{if } \sum \lambda_i \in \text{Coroot lattice.} \end{cases}$$

<u>Recall</u> $\mathrm{Gr}_\lambda \subset \mathrm{Gr}$ has $\dim = \langle 2\rho, \lambda \rangle$, $(\lambda = \text{dom cowt}, \ 2\rho = \sum_{\alpha \text{ pos coroot}} \alpha)$

It suggests

$$\begin{tikzcd} \mathrm{Sht}_G^{(1,\dots,r),\le\lambda} \arrow[d] & \sim & \prod_{i=1}^{r} \mathrm{Gr}_{\lambda_i} \\ X^r & & \end{tikzcd}$$

at least fiberwise same
(& w/ equal dim).

Fix $\lambda_1 = (1, 0, \dots, 0)$, $\lambda_2 = (0, \dots, 0, -1)$

<u>Thm</u> (Varshavsky) Locally for étale top,

$$\mathrm{Sht}_G^{(1,\dots,r),\le\lambda} \sim \left(\prod_{i=1}^{r} \mathrm{Gr}_{\le\lambda_i}\right) \times X^r.$$

$\Rightarrow$ $\mathrm{Sht}_G^{(1,\dots,r),\le\lambda}$ & $\prod_{i=1}^{r} \mathrm{Gr}_{\le\lambda_i}$ are not equi-dim'l,

but they have the same singularities.

Special case Let $\lambda_i$ minuscule.

$$\rightsquigarrow \mathrm{Gr}_{\lambda_i} = \mathrm{Gr}_{\leq \lambda_i} \cong G/P_{\lambda_i} \text{ smooth}$$

$$\Rightarrow \mathrm{Sht}_G^{(1,\dots,r), \leq \underline{\lambda}} \text{ is smooth over } X^r.$$

## §2 Non-iterated version of Sht

$I$ fin set.
$$\mathrm{Sht}_G^I = \left\{ \begin{array}{l} (x_i)_{i\in I}, \text{ with} \\ \mathcal{E} \overset{\alpha}{\dashrightarrow} {}^\tau\mathcal{E}, \text{ s.t. } \alpha|_{X \setminus \cup \Gamma(x_i)} \text{ isom} \end{array} \right\}$$

(do not order the legs)

Bounded ver $\underline{\lambda} = (\lambda_i)_{i\in I}$, take $I \xrightarrow{\sim} \{1,\dots,r\}$ (i.e. choosing an ordering on I)

Define
$$\mathrm{Sht}_G^{I, \leq \underline{\lambda}} := \mathrm{im}(\underbrace{\mathrm{Sht}_G^{(1,\dots,r), \leq \underline{\lambda}} \longrightarrow \mathrm{Sht}_G^I}_{\text{an isom away from diagonal (restr. to } X^r_{disj})})$$