Introduction to shtukas and their moduli (3/3)

Zhiwei Yun

July 14

Resume on For $G=GL_n$, $\lambda_1=(1,0,\dots,0)$, $\lambda_2=(0,\dots,0,-1)$, $x_1,x_2\in X^2$.

$$\begin{tikzcd}
Sht_G^{(1,2),\le\lambda} \arrow[d, "I\simeq\{1,2\}"] & \mathcal{E}_0 \arrow[r, hook', leftarrow, "x_1"] & \mathcal{E}_1 \arrow[r, hook, "x_2"] & {}^\tau\mathcal{E}_0 \\
Sht_G^{I,\le\lambda} & & &
\end{tikzcd}$$

$\mathcal{E}_0$: rank n

$\mathcal{E}_0 \overset{\alpha}{\dashrightarrow} {}^\tau\mathcal{E}_0$ with simple zero at x_2, simple pole at x_1

- $x_1\neq x_2$: $\mathcal{E}_1$ arises from $\mathcal{E}_0 \dashrightarrow {}^\tau\mathcal{E}_0$ uniquely $\Big\}$ $\mathcal{E}_1=\mathcal{E}_0\cap{}^\tau\mathcal{E}_0$.
- $x_1=x_2$: $\begin{cases} \alpha \text{ is not } \cong: & \text{unique } \mathcal{E}_1 \\ \alpha \text{ is } \cong: & \mathbb{P}^{n-1}\text{-choices of } \mathcal{E}_1 \end{cases}$

There are two important structures on moduli of shtukas:

Hecke corr & partial Frob.

§1 Hecke correspondences

$x\in|X| \rightsquigarrow K_x g K_x$, $K_x=G(\mathcal{O}_x)$.

$$\begin{tikzcd}
& Sht_{G,K_x\cap gK_xg^{-1}} \arrow[dl] \arrow[dr] & \\
Sht_G & & Sht_G
\end{tikzcd}
\qquad
\begin{tikzcd}
& C(K_xgK_x) = Sht^{I,\cdots}_{G,K_x\cap gK_xg^{-1}} \arrow[dl] \arrow[dr, "\text{forg}((-)\cdot g)"] \arrow[r] & (X\setminus x)^n \\
Sht_G^{I,\cdots} & & Sht_G^{I,\cdots}
\end{tikzcd}$$

Sht^I_{G,K'_x}, $K'_x\underset{\text{fin index}}{\subset} G(\mathcal{O}_x)=K_x$.

(level grp: $K'_x=\mathbb{K}'_x(k_x)$, $\mathbb{K}'_x\subset G[[t_x]]$ of fin codim.)

Define
$$Sht^I_{G,K'_x}=\left\{\begin{array}{l} x_i\in X\setminus x \text{ away from } x, \text{ with} \\ \mathcal{E}_i \text{ has } K'_x\text{-level at } x, \\ \text{same diagram respect } K'_x\text{-level} \end{array}\right\}$$

Typically, can take $K_x' = \mathrm{Iw}_x$.

In GL_n-case, $\mathcal{E}_i$ full flag of $\mathcal{E}_i|_x$.

Both maps $C(K_x g K_x) \longrightarrow \mathrm{Sht}_G^I$ are finite étale.

(fibers $\cong K_x / K_x \cap g K_x g^{-1}$.)

Composing, $H_x = C_c(G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)) \longrightarrow \prod \mathrm{Corr}_{f.ét}(\mathrm{Sht}_G^{I,\le\lambda}|_{(X\setminus x)^r})$

§2 Partial Frob

$$\begin{tikzcd}
\mathrm{Sht}_G^{(1,\dots,r),\le\lambda} \arrow[r, "\mathrm{Fr}_1"] \arrow[d] & \mathrm{Sht}_G^{(2,\dots,r,1)\le\lambda} \arrow[d] \\
X^r \arrow[r] & X^r \\
(x_1,\dots,x_r) \arrow[r, mapsto] & (x_2,\dots,x_r,\mathrm{Fr}(x_1))
\end{tikzcd}$$

$\mathrm{Fr}_1 : (\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \boxed{\mathcal{E}_r \cong {}^\tau\mathcal{E}_0})$

$\longrightarrow (\mathcal{E}_1 \dashrightarrow \mathcal{E}_2 \dashrightarrow \cdots \dashrightarrow \boxed{\mathcal{E}_r \cong {}^\tau\mathcal{E}_0} \dashrightarrow {}^\tau\mathcal{E}_1)$

$\rightsquigarrow (\mathrm{Fr}_1)^r = \mathrm{Fr}_{\mathrm{Sht}_G^{(1,\dots,r)}}$, Fr_1 is a homeomorphism (under ét top).

Let $\mathcal{M}_{G,K}$ = geom generic fiber of some Shimura var.

$\Gamma_F \circlearrowright \varinjlim_K H^*(\mathcal{M}^{\mu}_{G,K}) \circlearrowleft G(\mathbb{A}_f)$

$\mathrm{Gal}(\bar{F}/F) = \Gamma_F$

Same role as λ

$\Gamma_{F^{(r)}} \circlearrowright \varinjlim_K H^*(\mathrm{Sht}_{G,K}^{I,\le\lambda}) \circlearrowleft G(\mathbb{A})$ where $X^r \xrightarrow{pr_i} X \rightsquigarrow k(X^r) = F^{(r)}$.

func field of tr.deg $= r$. generic pt $\in X^r$ by Hecke corr.

V. Lafforge: $W_{F^{(r)}}$-action factors through $(W_F)^r$.

(using partial Fr)

§3 IC Sheaves

$\mathrm{Sht}_G^{I,\leq\lambda}$ not smooth in general.

$\rightsquigarrow$ It is natural to consider IC.

IC_λ = intersection complex on $Gr_{\leq\lambda}$, $Hk^{I,\leq\underline{\lambda}}$, $\mathrm{Sht}^{I,\leq\underline{\lambda}}$.

Have $\mathrm{Sht}_G^{(1,\dots,r),\leq\underline{\lambda}} \longrightarrow Hk_G^{(1,\dots,r),\leq\underline{\lambda}} \longrightarrow \prod_{i=1}^{r} (QGr_{\leq\lambda_i}/\mathrm{Aut}(D))^r$

records rel positions $(\mathcal{E}_{i-1} \overset{\alpha_i}{\dashrightarrow} \mathcal{E}_i)$

IC_λ for Sht is the pullback of $\boxtimes_{i=1}^{r} IC_{\lambda_i}$

Def'n I fin set, $W \in \mathrm{Rep}(\hat{G}^I)$ fin dim'l.

$\rightsquigarrow$ $H(I,W)$ def'd as follows

"Cohom of moduli of shtukas".

Note $W = \bigoplus \left(\boxtimes_{i\in I} V_{\lambda_i}\right) \rightsquigarrow H(I,W) = \bigoplus H(I, \boxtimes V_{\lambda_i})$

external tensor

Reduce to let $W = \boxtimes_{i\in I} V_{\lambda_i}$, $\underline{\lambda} = (\lambda_i)_{i\in I}$.

Choose an ordering $I \simeq \{1,2,\dots,r\}$

$$\begin{array}{cc} \mathrm{Sht}_G^{(1,\dots,r),\leq\underline{\lambda}} & IC_\lambda \\ \downarrow \pi & \\ X^r & \end{array}$$

$\rightsquigarrow$ $\mathcal{H}(I, \boxtimes V_{\lambda_i}) := R\pi_! IC_\lambda \in \mathrm{Ind}(D_c(X^r))$.

$H(I, \boxtimes V_{\lambda_i}) :=$ geom generic fiber of $\mathcal{H}$.

Fact $\mathcal{H}(I,W)$ is indep of ordering on I (canonically).

(can define it by $\mathrm{Sht}_G^{I,\leq\underline{\lambda}}$, $IC_{\underline{\lambda}}$)

Beilinson–Drinfeld Gr.

§4 Geometric Satake

$\mathrm{Perv}_c(QGr)$. supp on finitely many QGr_λ.

$$\mathrm{Perv}_c(\mathcal{G}r) \xleftarrow[\sim]{\otimes} \mathrm{Rep}(\hat{G}, \bar{\mathbb{Q}}_\ell) \leftarrow \text{a tensor cat.}$$

$$IC_\lambda \longleftrightarrow V_\lambda$$

About $\mathcal{H}(I,W)$

(1) Hecke action:

$$\begin{tikzcd} & C(K_x g K_x) \arrow[dl, "\text{fin ét}"'] \arrow[dr, "\text{fin ét}"] & \\ \mathrm{Sht}_G^{I,\leq\lambda}\big|_{(X\setminus x)^r} & & \mathrm{Sht}_G^{I,\leq\lambda}\big|_{(X\setminus x)^r}. \end{tikzcd}$$

$\rightsquigarrow$ induces $\mathcal{H}(I,W) \longrightarrow \mathcal{H}(I,W)$ restr to $(X\setminus x)^r$.
$\qquad\qquad\quad \uparrow$
$\qquad\qquad \boxtimes V_{\lambda_i}$

$\rightsquigarrow$ $H_x \circlearrowright \mathcal{H}(I,W)\big|_{(X\setminus x)^r}$ $\rightsquigarrow$ $\bigotimes'_{x\in|X|} H_x \circlearrowright H(I,W)$.

Spherical Hecke alg

(2) Partial Frob $\circlearrowright$ $H(I,W)$

$$\begin{tikzcd} \mathrm{Fr}_1 : \mathrm{Sht}_G^{(1,\dots,r),\leq\lambda} \arrow[r] \arrow[d] & \mathrm{Sht}_G^{(2,\dots,r,1),\leq\lambda} \arrow[d] \\ X^I \arrow[r, "\text{forget the ordering}", "{(\mathrm{Fr},\mathrm{id},\dots,\mathrm{id}) = \mathrm{Fr}_{X,1}}"'] & X^I \end{tikzcd} \qquad (I = \{1,\dots,r\})$$

$\rightsquigarrow$ $H(I,W) \xleftarrow{\sim} \mathrm{Fr}_{X,1}^* H(I,W)$.

For $i\in I$, $\phi_i : \mathrm{Fr}_{\{i\}}^* H(I,W) \xrightarrow{\sim} \mathcal{H}(I,W)$

different $i\in I$: (ϕ_i) commute.

$\prod \phi_i$ = Weil's str on $\mathcal{H}(I,W)$.

(3) Factorization: e.g.

$$\begin{tikzcd} \mathrm{Sht}^{\{*\},\leq\lambda_1+\lambda_2} \arrow[r, hook] \arrow[d] & \mathrm{Sht}^{I,\leq(\lambda_1,\lambda_2)} \arrow[d] \\ X \arrow[r, hook, "\Delta"] & X^2 \end{tikzcd} \qquad (I=\{1,2\}).$$

Fact $\Delta X \overset{\Delta}{\hookrightarrow} X^I \Rightarrow \mathcal{H}(*, W|_{\Delta\hat{G}}) \simeq \Delta^* \mathcal{H}(I,W)$ $\quad (W \in \mathrm{Rep}(\hat{G}^I))$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
pf. decomposable $\leftsquigarrow$ $\boxtimes V_{\lambda_i}$

When $I \overset{\theta}{\twoheadrightarrow} J$, $X^J \overset{\Delta_\theta}{\hookrightarrow} X^I$, $\hat{G}^J \overset{\Delta_\theta}{\hookrightarrow} \hat{G}^I$.

$$\Rightarrow \mathcal{H}(J, W|_{\Delta_\theta(\hat{G}^J)}) \cong \Delta_\theta^* \mathcal{H}(I, W).$$