

Introduction to shtukas and their moduli (3/3)

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Resume on For $G = GL_n$, $\lambda_1 = (1, 0, \dots, 0)$, $\lambda_2 = (0, \dots, 0, -1)$, $x_1, x_2 \in X^2$,

$$\text{Sht}_G^{(\lambda_1, \lambda_2), \Sigma} \quad \begin{array}{c} \textcircled{\Sigma} \xrightarrow{x_1} \Sigma_1 \xrightarrow{x_2} \Sigma_0 \\ \uparrow \\ \text{rank } n \end{array}$$

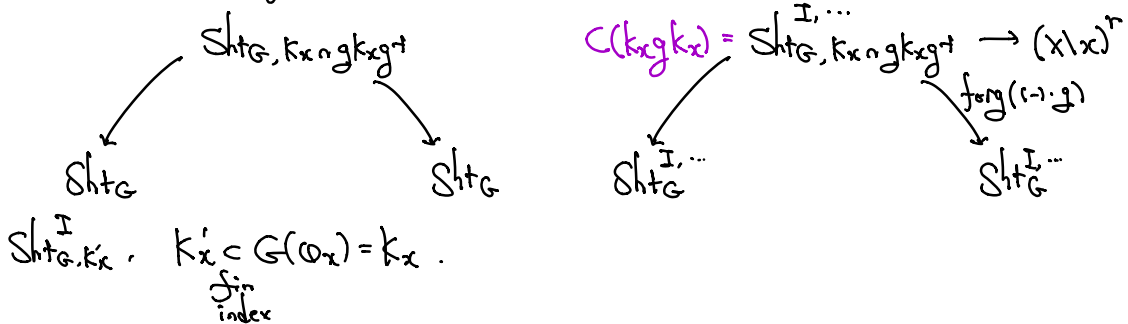
$$\downarrow I \approx \{1, 2\} \quad \Sigma_0 \xrightarrow{-\alpha} \Sigma_1 \text{ with } \begin{array}{l} \text{Simple zero at } x_2 \\ \text{Simple pole at } x_1 \end{array}$$

- $x_1 \neq x_2$: Σ_1 arises from $\Sigma_0 \dashrightarrow \Sigma_1$ uniquely $\left\{ \begin{array}{l} \Sigma_1 = \Sigma_0 \circ \tau \Sigma_0 \end{array} \right.$
- $x_1 = x_2$: $\left\{ \begin{array}{l} \alpha \text{ is not } \cong : \text{ unique } \Sigma_1 \\ \alpha \text{ is } \cong : \mathbb{P}^{n-1} \text{-choices of } \Sigma_1 \end{array} \right.$

There are two important structures on moduli of shtukas:
Hecke corr & partial Frob.

§1 Hecke correspondences

$$x \in |X| \rightsquigarrow K_x \supset K_x, \quad K_x = G(\mathcal{O}_x)$$



(level grp: $K'_x = K'_x(K_x)$, $K'_x \subset G[\![t_x]\!] \text{ of fin codim.}$)

Define

$$\text{Sht}_{G, K'_x}^I = \left\{ \begin{array}{l} x_i \in X \setminus X \text{ away from } x, \text{ with } \left\{ \begin{array}{l} \Sigma_i \text{ has } K'_x\text{-level at } x, \\ \text{same diagram respect } K'_x\text{-level} \end{array} \right. \end{array} \right.$$

Typically, can take $K'_i = I_{W_i}$.

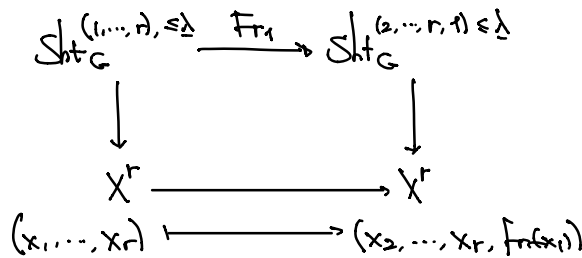
In GLn case, \mathcal{E}_i full flag of $\mathcal{E}_i|_X$.

Both maps $C(K_x \times K_x) \rightarrow \text{Sh}_G^I$ are finite étale.

(fibers $\cong K_x / K_x \cap gK_x g^{-1}$.)

Composing, $H_X = C_c(G(\mathcal{O}_X) \backslash G(\mathbb{F}_X) / G(\mathcal{O}_X)) \rightarrow \mathbb{Z} \text{Corr}_{f.\text{ét}}(\text{Sh}_G^{I, S, \lambda} / (X|X)^r)$

\mathcal{E}_2 Partial Frobenius



$$Fr_1: (\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \dashrightarrow \dots \dashrightarrow \boxed{\Sigma_r \cong \mathcal{E}_0})$$

$$\rightarrow (\mathcal{E}_1 \dashrightarrow \mathcal{E}_2 \dashrightarrow \dots \dashrightarrow \boxed{\Sigma_r \cong \mathcal{E}_0} \dashrightarrow \mathcal{E}_1)$$

$\hookrightarrow (Fr_1)^r = Fr_{\text{Sh}_G^{(1, \dots, r), S, \lambda}}$, Fr_1 is a homeomorphism (under ét top).

Let $M_{G, K} = \text{geom generic fiber of some Shimura var.}$

$$\Gamma_F \subset \varinjlim_K H^*(M_{G, K}^{\mu}) \hookrightarrow G(\mathbb{A}_F)$$

$\text{Gal}(\bar{F}/F)$ same role as λ

$$\Gamma_{F^{(n)}} \subset \varinjlim_K H^*(\text{Sh}_{G, K}^{I, S, \lambda}) \hookrightarrow G(\mathbb{A}) \quad \text{where } X^r \xrightarrow{pr_i} X \mapsto k(X^r) = F^{(n)}$$

$\Gamma_{F^{(n)}}$ ↑ func field of tr. deg = r. ↑ generic pt $\in X^r$ ↑ by Hecke corr.

v. Lafforge: $W_{F^{(n)}}$ -action factors through $(W_F)^r$
(using partial Fr)

§3 IC Sheaves

$\text{Sh}_G^{I, \epsilon_\lambda}$ not smooth in general.

↳ It is natural to consider IC.

$\text{IC}_\lambda =$ intersection complex on G_{r, ϵ_λ} , $\text{Hk}^{I, \epsilon_\lambda}$, $\text{Sh}_G^{I, \epsilon_\lambda}$.

Have $\text{Sh}_G^{(i_1, \dots, i_r), \epsilon_\lambda} \longrightarrow \text{Hk}_G^{(i_1, \dots, i_r), \epsilon_\lambda} \xrightarrow{\uparrow} \prod_{i=1}^r (\mathbb{Q}G_{r, \epsilon_{\lambda_i}} / \text{Aut}(\mathcal{D}))^r$

records rel positions $(\epsilon_{i-1} \dashrightarrow \epsilon_i)$

IC_λ for Sh_G is the pullback of $\prod_{i=1}^r \text{IC}_{\lambda_i}$

Def'n I fin set, $W \in \text{Rep}(\hat{G}^I)$ fin diml.

↳ $\mathcal{H}(I, W)$ def'd as follows

"Cohom of moduli of strukas".

Note $W = \bigoplus_{i \in I} (\boxtimes_{i \in I} V_{\lambda_i}) \rightsquigarrow \mathcal{H}(I, W) = \bigoplus \mathcal{H}(I, \boxtimes V_{\lambda_i})$

external tensor

Reduce to let $W = \boxtimes_{i \in I} V_{\lambda_i}$, $\underline{\lambda} = (\lambda_i)_{i \in I}$.

Choose an ordering $I \simeq \{1, 2, \dots, r\}$

$\text{Sh}_G^{(i_1, \dots, i_r), \epsilon_\lambda} \quad \text{IC}_\lambda$

$\pi \downarrow$
 X^r

↳ $\mathcal{H}(I, \boxtimes V_{\lambda_i}) := \mathbb{R}\pi_! \text{IC}_\lambda \in \text{Ind}(\mathcal{D}_c(X^r))$.

$\mathcal{H}(I, \boxtimes V_{\lambda_i}) :=$ geom generic fiber of \mathcal{H} .

Fact $\mathcal{H}(I, W)$ is indep of ordering on I (canonically).

(can define it by $\text{Sh}_G^{I, \epsilon_\lambda}$, IC_λ)

Beilinson-Drinfeld Gr.

§4 Geometric Satake

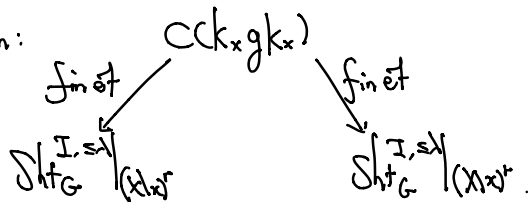
$\text{Perv}_c(\mathbb{Q}\text{Gr})$. Supp on finitely many $\mathbb{Q}\text{Gr}_\lambda$.

$$\text{Per}_{\mathbb{C}}(\mathcal{QGr}) \xleftarrow{\cong} \text{Rep}(\widehat{G}, \overline{\mathbb{Q}}_l) \leftarrow \text{a tensor cat.}$$

$$\text{IC}_{\lambda} \xleftarrow{\quad} V_{\lambda}$$

About $\mathcal{H}(\mathbb{I}, W)$

(1) Hecke action:



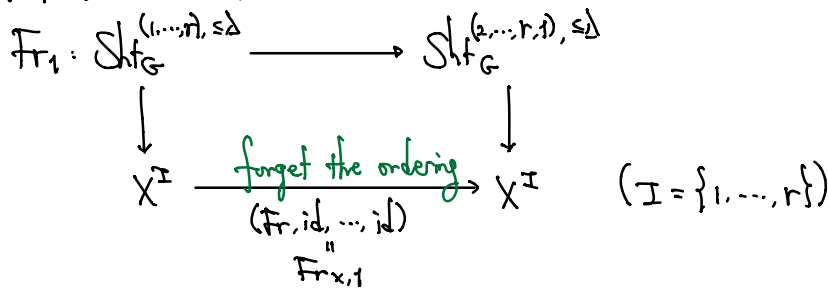
\hookrightarrow induces $\mathcal{H}(\mathbb{I}, W) \rightarrow \mathcal{H}(\mathbb{I}, W)$ restr to $(X|X)^r$.

$$\uparrow \boxtimes V_{\lambda}$$

$$\hookrightarrow H_x \subset \mathcal{H}(\mathbb{I}, W)|_{(X|X)^r} \hookrightarrow \bigotimes_{x \in (X|X)} H_x \subset H(\mathbb{I}, W).$$

Spherical Hecke alg

(2) Partial Frob $\subset H(\mathbb{I}, W)$



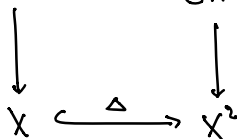
$$\hookrightarrow H(\mathbb{I}, W) \xleftarrow{\sim} \text{Fr}_{x,1}^* H(\mathbb{I}, W).$$

$$\text{For } i \in \mathbb{I}, \phi_i: \text{Fr}_{i,1}^* H(\mathbb{I}, W) \xrightarrow{\sim} H(\mathbb{I}, W)$$

different $i \in \mathbb{I}$: (ϕ_i) commute.

$\prod \phi_i = \text{Weil's str on } \mathcal{H}(\mathbb{I}, W).$

(3) Factorization: e.g. $\text{Sh}_G^{(1,2), s, \lambda + k_2} \xrightarrow{\quad} \text{Sh}_G^{\mathbb{I}, s, \lambda_2} \quad (\mathbb{I} = \{1, 2\}).$



$$\text{Fact } \Delta X \xrightarrow{\Delta} X^{\mathbb{I}} \Rightarrow \mathcal{H}(*, \mathbb{N}/\Delta \widehat{G}) \simeq \Delta^* \mathcal{H}(\mathbb{I}, W) \quad (W \in \text{Rep}(\widehat{G}^{\mathbb{I}})).$$

pt. decomposable $\leftarrow \boxtimes V_{\lambda}$

$$\text{When } I \xrightarrow{\theta} J, \quad X^J \xrightarrow{\Delta_\theta} X^I, \quad \widehat{G}^J \xrightarrow{\Delta_\theta} \widehat{G}^I.$$
$$\Rightarrow \mathcal{H}(J, w|_{\Delta_\theta(\widehat{G}^J)}) \cong \Delta_\theta^* \mathcal{H}(I, w).$$