# Prismatic cohomology (1/4)

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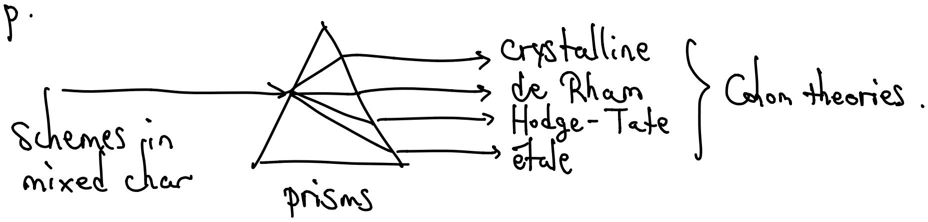
Fix prime $p$.




$X/\mathbb{Z}_p$ sm $p$-adic formal sch $\rightsquigarrow R\Gamma_{crys}(X_s/\mathbb{Z}_p)$ on special fiber

$X_s := X \times_{\mathrm{Spf}\,\mathbb{Z}_p} \mathrm{Spec}\,\mathbb{F}_p$

$R\Gamma_{dR}(X/\mathbb{Z}_p) = R\Gamma(X, \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/\mathbb{Z}_p} \xrightarrow{d} \cdots)$

$R\Gamma_{HT}(X)$ twisted form of $R\Gamma(X, \mathcal{O}_X \xrightarrow{0} \Omega^1_{X/\mathbb{Z}_p} \xrightarrow{0} \cdots)$

$R\Gamma_{et}(X_\eta, \mathbb{Z}_p)$, $X_\eta$ = rigid geom fiber.

## § Prismatic cohom in char 0

$k$ perfect field of char 0. $A := k[[t]]$. $R$ $k$-alg.

**Def'n** (prismatic site in char 0)

$(R/A)^{op}_{\Delta}$ with · objects $(R \xrightarrow{\iota} S/t \leftarrow S)$

with $S$ = $t$-complete $t$-torsion-free $A$-alg

& $R \xrightarrow{\iota} S/t$ map of $k$-algs.

· morphs $(R \xrightarrow{\iota} S/t \leftarrow S) \longrightarrow (R \xrightarrow{\iota'} S'/t \leftarrow S')$

given by $\alpha: S \to S'$ map of $A$-algs

s.t. $\alpha \circ \iota = \iota'$.

· covers given by isoms.

Set $\mathcal{O}_{\Delta}(R \to S/t \leftarrow S) = S$, $\bar{\mathcal{O}}_{\Delta}(R \to S/t \leftarrow S) = S/t$.
reduced ver.

$$R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) = R\varprojlim_{(R\to S/t\leftarrow S)} S \quad \text{derived inverse limit.}$$

<u>Theorem</u> (Wassmuth) $R$ sm over $k$.

(1) $H^i((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \Omega^i_{R/k}$, $i \geq 0$ canonically

(2) Assume $\tilde{R}$ $t$-complete $t$-torsion-free lift of $R$ to $A$.

Then $R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta}) = [\tilde{R} \xrightarrow{t\cdot d} \tilde{\Omega}^1_{\tilde{R}/A} \xrightarrow{t\cdot d} \tilde{\Omega}^2_{\tilde{R}/A} \longrightarrow \cdots]$

<u>Sketch of pf</u> <u>Step1</u> Comparison map in (1).

Have ses $0 \longrightarrow \bar{\mathcal{O}}_{\Delta} \xrightarrow{t} \mathcal{O}_{\Delta}/t^2 \longrightarrow \bar{\mathcal{O}}_{\Delta} \longrightarrow 0$

$\rightsquigarrow$ get
$$\begin{array}{ccc} H^0((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) & \xrightarrow{\beta} & H^1((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \\ \uparrow {\scriptstyle 2_{abs}} \text{ (a canonical map by def'n)} & & \\ R & & \end{array}$$

Now $H^*((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta})$ is a commutative graded alg

& $\beta \circ 2_{abs}$ is a $k$-linear derivation.

$\Rightarrow$ get a unique ext'n $\alpha_{HT} : \Omega^*_{R/k} \longrightarrow H^*((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta})$.

$\rightsquigarrow$ reduce (1) to (2).

<u>Step2</u> A formula for $R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$.

<u>Lem</u> $(R/A)_{\Delta}$ has fiber products (via $\hat{\otimes}$) and non-empty product.

To get products:

<u>Def'n</u> $T$ $t$-complete, $t$-torsion-free $A$-alg. $J \subseteq T$ any ideal, $t \in J$.

The prismatic envelope $D_J(T)$ of $T$ in $J$ is $T[\frac{j}{t} \mid j \in J]^{\wedge}_t$.

(t-adic completion).

**Note** $S$ $t$-complete, $t$-torsion-free $A$-alg.

Then $\mathrm{Hom}((T,J),(S,(t))) \simeq \mathrm{Hom}((D_J(T),(t)),(S,(t)))$.

Now, let $(R \xrightarrow{\iota} S/t \leftarrow S)$, $(R \xrightarrow{\iota'} S'/t \leftarrow S')$.

Set $T := S \hat{\otimes}_A S'$, $J = \ker(T \to T/t = S/t \otimes_R S'/t \longrightarrow S/t \otimes_R S'/t)$.

Then $(R \xrightarrow{\iota \otimes \mathrm{Id}} T/t \longrightarrow D_J(T)/t \longleftarrow D_J(T))$ represents the product.

Thus, $R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta)$ can be calculated as follows.

- Pick $P$ to be any $t$-complete smooth $A$-alg with surj $P \twoheadrightarrow R$ & kernel $= J$.
- Set $D := D_J(P)$. Then $(R \to D/t \leftarrow D) \in (R/A)_\Delta$ covers the final object of $\widetilde{(R/A)_\Delta}$.

$\rightsquigarrow$ $R\Gamma((R/A)_\Delta, \mathcal{O}_\Delta) = (D(0) \longrightarrow D(1) \longrightarrow D(2) \to \cdots)$ complex of rings

where $D(i) = (i+1)$th fold product of $D$ in $(R/A)_\Delta$.

**Note** $D(n) =$ prism envelope of $P^{\hat{\otimes}_A (n+1)}$

and $J_n = \ker(P^{\otimes n+1} \twoheadrightarrow R) = (t, x_1, \cdots, x_r)$

locally generated by regular sequence.

$$\Rightarrow D(n) = P^{\hat{\otimes}(n+1)}\left[\frac{x_1}{t}, \cdots, \frac{x_r}{t}\right]^\wedge_t.$$

$$\& \quad D(n)/t \simeq R\left[\frac{x_1}{t}, \cdots, \frac{x_r}{t}\right].$$

**Step 3** $t$-de Rham complexes.

Indeed,
$$\begin{tikzcd} P \arrow[r, "t \cdot d"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \hat{\Omega}^1_{P/A} \hat{\otimes}_P D \\ D \arrow[ur, dashed, "\exists ! \nabla^t"'] & \end{tikzcd}$$
$\nabla^t\left(\frac{j}{t}\right) := dj$ for $j \in J$.

Step 4 Finish the argument.

Set $P(n) = P^{\hat{\otimes}_A (n+1)}$. Set $K^{a,b} := \hat{\Omega}^b_{P(a)/A} \hat{\otimes}_{P(a)} D(a)$.

$$\begin{tikzcd} \vdots & \\ \hat{\Omega}^1_{P(0)/A} \hat{\otimes}_P D \arrow[u] \arrow[r] & \cdots \\ D(0) \arrow[u, "\nabla^t"] \arrow[r] & D(1) \arrow[u] \arrow[r] & D(2) \end{tikzcd}$$

Rather formal: $K^{\bullet,b}$ acyclic for $b>0$.

$\rightsquigarrow \mathrm{Tot}(K^{\bullet,\bullet}) \simeq (D(0) \to D(1) \to \cdots)$.

Claim each of the cosimplicial str maps $D(0) \to D(n)$ induces a quasi-isom $K^{0,\bullet} \to K^{n,\bullet}$.

$\rightsquigarrow \mathrm{Tot}(K^{\bullet,\bullet}) \simeq R\varprojlim_{\Delta} K^{n,\bullet} = K^{0,\bullet}$

$\simeq$ (assume $\overset{D}{\overset{\shortparallel}{P}} = \tilde{R}$ & $\tilde{R}/t = R$.)

$(\tilde{R} \xrightarrow{t\cdot d} \hat{\Omega}^1_{\tilde{R}/A} \xrightarrow{t\cdot d} \cdots)$

Lem $S$ $t$-complete, $t$-torsion-free $A$-alg.

$P = S[x_1, \dots, x_r]^{\wedge}_t$, $J = (t, x_1, \dots, x_r)$, $D := D_J(P)$.

$\Rightarrow 0 \to S \to D \xrightarrow{\nabla^t} \hat{\Omega}^1_{P/S} \hat{\otimes}_P D \to \cdots$ exact.

proof Reduce mod $t$.

$D/t \simeq S/t[\frac{x_1}{t}, \dots, \frac{x_r}{t}]$, $\nabla^t(\frac{x_i}{t}) = dx_i$.

$\rightsquigarrow$ apply usual Poincaré lem for poly rings in char 0. $\square$

## § Prismatic cohom in char p

Fix a prime $p$. All rings are $p$-adic complete.

Def'n A $\delta$-ring is a ring $A$ with a map

$\delta := \delta_A : A \longrightarrow A$ (of sets) s.t.

(i) $\delta(0) = 0$, $\delta(1) = 0$,

(ii) $\delta(x \cdot y) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\delta(y)$.

(iii) $\delta(x+y) = \delta(x) + \delta(y) + \frac{1}{p}(x^p + y^p - (x+y)^p)$.

Given $\delta$-ring $A$, set $\varphi := \varphi_A : A \to A$, $x \mapsto x^p + p \cdot \delta(x)$.

<u>Lemma</u> (1) $A$ $\delta$-ring $\Rightarrow$ $\varphi_A$ is a lift of Frobenius.

(2) $A$ $p$-torus, $\varphi : A \to A$ lift of Frobenius.

$\delta(x) := \frac{1}{p}(\varphi(x) - x^p)$ is a $\delta$-str on $A$.

Fix a perfect field $k$, $R$ $k$-alg.

<u>Def'n</u> (prismatic site in char $p$)

$(R/W(k))_{\Delta}$ has

- objects $(R \xrightarrow{\iota} A/p \leftarrow A)$

  with $A$ $p$-complete $p$-torsion-free $\delta$-ring over $W(k)$.
- morphs : morphs of $\delta$-rings $/W(k)$ compatible with $\iota$
- Covering : topology $A \to B$ cover if

  $A/p \to B/p$ is faithfully flat.

Set $\mathcal{O}_{\Delta}(R \to A/p \leftarrow A) = A$, $\bar{\mathcal{O}}_{\Delta}(R \to A/p \leftarrow A) = A/p$.

<u>Theorem</u> (Bhatt-Scholze, Ogus) $R$ sm $/k$.

(1) $\varphi^*_{W(k)} R\Gamma((R/W(k))_{\Delta}, \mathcal{O}_{\Delta}) \simeq R\Gamma_{crys}(R/W(k))$.

(2) $H^*((R/W(k))_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \Omega^*_{R/k}$ canonically.

(3) If $\tilde{R}$ $p$-completely sm $\delta$-lift of $R$ to $W(k)$,

then $R\Gamma((R/W(k))_{\Delta}, \mathcal{O}_{\Delta}) \simeq [\tilde{R} \xrightarrow{p \cdot d} \hat{\Omega}^1_{\tilde{R}/W(k)} \xrightarrow{p \cdot d} \cdots]$ $\varphi$-equiv.

__Rmk__ * $\Omega^*_{R/k} \longrightarrow H^*((R/W(k))_\Delta, \overline{\mathcal{O}}_\Delta)$ can be constructed as before.

In (3).

$$\begin{tikzcd}
R\Gamma((R/W(k))_\Delta, \mathcal{O}_\Delta) \arrow[d] \arrow[r, phantom, "\simeq"] & \left[\tilde{R}\right. \arrow[r, "d"] \arrow[d, "\varphi_{\tilde{R}}"] & \frac{1}{p}\hat{\Omega}^1_{\tilde{R}/W(k)} \arrow[r, "d"] \arrow[d, "\varphi_{\tilde{R}}"] & \frac{1}{p^2}\hat{\Omega}^2_{\tilde{R}/W(k)} \arrow[r, "d"] & \left.\cdots\right] \\
\varphi_{W(k),*} R\Gamma((R/W(k))_\Delta, \mathcal{O}_\Delta) \arrow[r, phantom, "\simeq"] & \varphi_{W(k),*}\left[\tilde{R}\right. \arrow[r, "d"] & \hat{\Omega}^1_{\tilde{R}/W(k)} \arrow[r, "d"] & \left.\cdots\right].
\end{tikzcd}$$

Modulo $p$ $\rightsquigarrow$ reduces to Cartier isom

$$\Omega^1_{R^{(1)}/k} \xrightarrow{\sim} H^i(\Omega^*_{R/k})$$

for $R^{(1)} = R \otimes_{k,\varphi} k$.

__Some ingredients in the pf of (1)__

__Lemma__ $H: \{\delta\text{-rings}\} \longrightarrow \{\text{rings}\}$

has a left adjoint $F$ & a right adjoint $G$.

In fact, $G(R) = W(R)$ = $p$-typical Witt vectors of $R$.

& $\mathbb{Z}\{x\} := F(\mathbb{Z}[x]) = \mathbb{Z}[x, \delta(x), \delta^2(x), \dots]$.

__Prop'n__ Given $p$-complete, $p$-torsion-free $\delta$-$W(k)$-alg $P$

and $J = (p, x_1, \dots, x_r)$ via a regular seq,

then $D := P\{\frac{x_1}{p}, \dots, \frac{x_r}{p}\}^{\wedge}_p$ is the prism envelope of $P$ in $J$.

Hom of $\delta$-pairs.

Also, $\mathrm{Hom}((P, J), (A, (p))) = \mathrm{Hom}((D, (p)), (A, (p)))$

$p$-complete $p$-torsion-free

__Key lemma__ $A$ $p$-torsion-free $\delta$-$\mathbb{Z}_p$-alg, $x \in A$ s.t. $\varphi(x)/p \in A$.

Then $x$ has all divided powers.

In fact, if $(p, x_1, \dots, x_r)$ is regular,

then $A\left\{\frac{\varphi(x_1)}{p}, \dots, \frac{\varphi(x_r)}{p}\right\}$ = PD-envelope of $(x_1, \dots, x_r)$ in $A$.

Sketch pf $\frac{\varphi(x)}{p} = \frac{x^p}{p} + \delta(x) \rightsquigarrow \gamma_p(x) = \frac{x^p}{p!} \in A$.

Claim $\gamma_{p^2}(x) = \underset{\mathbb{Z}_p}{\underset{\cap}{\text{unit}}} \cdot \frac{x^{p^2}}{p^{p+1}} \in A$

Thus $\underset{\underset{A}{\cap}}{\delta\left(\frac{x^p}{p}\right)} = \frac{1}{p}\left(\frac{\varphi(x)}{p} - \frac{x^{p^2}}{p^p}\right) = \underbrace{\frac{1}{p^2}(x^p + p\,\delta(x))^p}_{\underset{A}{\cap}} - \frac{x^{p^2}}{p^{p+1}}$.

b/c $p^{p-2}\left(\frac{x^p}{p} + \delta(x)\right)^p \in A$.

Finish by induction using

$\gamma_{k \cdot p}(y) = \text{unit} \cdot \gamma_k(\gamma_p(y))$. $\square$