Prismatic cohomology (2/4)

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§ General properties of (derived) prismatic cohom

Fix a prime p.

Def'n A *prism* is a pair (A, I) of a δ-ring A & an invertible ideal $I \subseteq A$ s.t.

(1) A is (derived) (p, I)-complete.

(2) $p \in (I, \varphi(I))$ ($\Leftrightarrow$ after some localization of A, $I = (d)$ with $\delta(d) \in A^\times$.)

Examples (1) A p-complete p-torsion-free, $I = (p)$

$\rightsquigarrow$ (A, I) prism "crystalline case".

Note: $\delta(p) = \frac{p - p^p}{p} = 1 - p^{p-1} \in A^\times$.

(2) "Breuil-Kisin case prism"

k perfect field, $A := W(k)[[u]]$, $\delta(u) = 0$
$\Updownarrow$
$\varphi(u) = u^p$.

$I = (E(u))$, $E(u) \in W(k)[u]$ Eisenstein polynomial.

Note $A/I \xrightarrow{\sim} \mathcal{O}_K$ via $u \mapsto \varpi$

for K p-adic field & ϖ a chosen uniformizer.

(3) "cyclotomic prism"

$A = \mathbb{Z}_p[[q-1]]$, $\varphi(q) = q^p$,

$I = ([p]_q = \frac{q^p - 1}{q-1} = 1 + q + \cdots + q^{p-1})$

$A/I \cong \mathbb{Z}_p[\zeta_p]$, $q \mapsto \zeta_p$.

Def'n (A,I) prism is called

(1) bounded, if A/I has bdd p^∞-torsion.

(2) orientable, if I principal.

(3) perfect, if $\varphi: A \xrightarrow{\sim} A$ isom.

(4) transversal, if A/I is p-torsion-free.

Def'n $(A,I) \to (B,J)$ is (faithfully) flat if

$A \to B$ is (p,I)-completely (faithfully) flat

i.e. $A/^{\mathbb{L}}(p,I) \longrightarrow B/^{\mathbb{L}}(p,I)$ (faithfully) flat.

derived reduction

Rmk $\cdot$ (A,I) perfect $\iff$ $A = W(R)$, R perfect $\mathbb{F}_p$-alg

and $I = (\xi)$, $\xi = [a_0] + p[a_1] + \cdots$

and R a_0-adically complete, $a_1 \in R^\times$.

$\cdot$ $\{\text{perfect prisms}\} \xleftrightarrow{\sim} \{\text{perfectoid rings}\}$

$(A,I) \longmapsto A/I$

$(A_{\inf}(T) = W(T^\flat), \ker\theta) \longleftarrow T$.

Def'n (prismatic site)

(1) ("relative") (A,I) bdd prism, R A/I-alg.

$(R/A)_\Delta$ has $\cdot$ objects $(R \xrightarrow{2} B/J \leftarrow B)$

(B,J) bdd prism over (A,I)

& 2 morph of A/I-algs.

$\cdot$ morphs: morphs of prisms compatible with 2 $/(A,I)$.

$\cdot$ Covers: faithfully flat maps of prisms.

Set $\mathcal{O}_{\Delta}(R \longrightarrow B/J \leftarrow B) = B$, $\bar{\mathcal{O}}_{\Delta}(R \to B/J \leftarrow B) = B/J$.

(2) ("absolute") Same without base prism (A, I).

$\rightsquigarrow$ Notation $(R)_{\Delta}$ instead of $(R/A)_{\Delta}$ or $(R/W(k))_{\Delta}$.

Lemma (Rigidity) $(A, I) \longrightarrow (B, J)$ morph of prisms.

Then $I \cdot B = J$.

proof Reduce to $I = (d)$, $J = (d')$, $d = u \cdot d'$ with $u \in B$.

$$\Rightarrow \underset{B^{\times}}{\underset{\cap}{\delta(d)}} = u^p \cdot \delta(d') + \underbrace{(d')^p \cdot \delta(u) + p \cdot \delta(u) \cdot \delta(d')}_{\in \operatorname{rad} B}$$

$$\Rightarrow u^p \in B^{\times} \Rightarrow u \in B^{\times}. \qquad \square$$

Cor $(R/A)_{\Delta}$, $(R)_{\Delta}$ have non-empty products calculated by prismatic envelopes.

Theorem (Bhatt-Scholze) R sm over A/I. (A, I) bdd prism.

$$R\Gamma_{\Delta}(R/A) := \underset{\substack{\circlearrowleft \\ \varphi}}{R\Gamma}((R/A)_{\Delta}, \mathcal{O}_{\Delta}) \in \mathrm{CAlg}(\underset{\substack{\uparrow \\ \text{derived } \infty\text{-cat of } A}}{\hat{D}(A)})$$

Then (1) ("crystalline compatibility")

If $I = (p)$, then

$$R\Gamma_{crys}(R/A) \simeq R\Gamma_{\Delta}(R/A) \otimes^{\mathbb{L}}_{A,\varphi} A.$$

(2) ("Hodge-Tate comparison")

$$H^*((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \Omega^*_{R/(A,I)} \underbrace{\{-*\}}_{\substack{\uparrow \\ (\cdot) \otimes_{A/I} (I/I^2)^{-*}}}.$$

$$(0 \to I/I^2 \to \mathcal{O}_{\Delta}/I^2 \longrightarrow \mathcal{O}_{\Delta}/I \longrightarrow 0).$$

(3) ("de Rham comparison")

$$R\Gamma_{dR}(R/(A/I)) \simeq (R\Gamma_{\Delta}(R/A) \hat{\otimes}^{\mathbb{L}}_{A,\varphi} A) \hat{\otimes}^{\mathbb{L}}_{A} A/I.$$

(4) ("étale cohomology")

Assume A perfect. Let X_η = generic fiber of $\mathrm{Spf}\, R$. Then

$$R\Gamma_{\acute{e}t}(X_\eta, \mathbb{Z}/p^n) \simeq (R\Gamma_{\Delta}(R/A) \otimes^{\mathbb{L}}_{A} A/p^n[\tfrac{1}{I}])^{\varphi=1}.$$

(5) $R\Gamma_{\Delta}(R/A)$ commutes with base changes in (A, I).

<u>Key technical input</u>

Assume $I=(d)$ and assume $P = (p,I)$-complete flat δ-A-alg

$J = (d, x_1, \dots, x_r) \subseteq P$ with $x_1, \dots, x_r$ (p,d)-completely regular seq.

Then the prismatic envelope $P\{\frac{J}{I}\}^\wedge$ of P in J is $P\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}^\wedge_{(p,I)}$.

Moreover, $P\{\frac{J}{I}\}^\wedge$ is (p,I)-completely flat over A

& commutes with base changes.

<u>Example</u> (q-de Rham cohom)

$A = \mathbb{Z}_p[[q-1]] \supseteq I = ([p]_q)$, $R = \mathbb{Z}_p\langle T^{\pm 1}\rangle$, $\varphi(q) = q^p$.

where $[n]_q = \frac{q^n - 1}{q-1} = 1 + q + \cdots + q^{n-1}$.

Set $R^{(1)} = R \otimes_{\mathbb{Z}_p} A/I \simeq \mathbb{Z}_p[\zeta_p]\langle T^{\pm 1}\rangle$ & $\tilde{R} = A\langle T^{\pm 1}\rangle$. $\varphi(T) = T^p$.

<u>Thm</u> (Bhatt-Scholze)

$\Delta_{R^{(1)}/A} \simeq$ q-de Rham complex of $\tilde{R}$ over A

$= [\tilde{R} \xrightarrow{\nabla_q} \tilde{R} \cdot d_q T]$ with $\nabla_q(f(T)) := \frac{f(qT) - f(T)}{qT - T} \cdot d_q T$.

(Check: $\nabla_q(T^n) = [n]_q \cdot T^{n-1} \cdot d_q T$.)

$\simeq [\widehat{\bigoplus_{n \in \mathbb{Z}}} A \cdot T^n \xrightarrow{[n]_q} \widehat{\bigoplus_{n \in \mathbb{Z}}} A \cdot T^n \cdot \underbrace{\frac{d_q T}{T}}_{= d_q \log T}]$.

Rmk * de Rham comparison follows from $n \equiv [n]_q \bmod q-1$.

* HT comparison is also explicit.

Note $\nabla_q(f \cdot g(T)) = f(q \cdot T) \cdot \nabla_q(g(T)) + g(T) \cdot \nabla_q(f(T))$. (quasi-chain rule).

Set $A_\infty := (\varinjlim_\varphi A)^\wedge = (\bigcup_{n \geq 1} \mathbb{Z}_p[\![q^{1/p^n}-1]\!])^\wedge_{(\varphi, I)} = A_{inf}(\mathbb{Z}_p^{cycl})$

$$q \longmapsto [\varepsilon], \quad \varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots)$$

Note $(q-1) \in L := (A_\infty/p)[\frac{1}{I}]$ invertible b/c $[p]_q = (q-1)^{p-1} \bmod p$.

$$\Rightarrow \quad \Delta_{R^{(1)}/A} \overset{\mathbb{L}}{\otimes}_A L \simeq [\widehat{\bigoplus_{n \in \mathbb{Z}}} L \cdot T^n \xrightarrow{q^n - 1} \widehat{\bigoplus_{n \in \mathbb{Z}}} L \cdot T^n \cdot \frac{d_q \log T}{q-1}] =: K^\bullet$$

Now (1) $\varphi(d_q T) = d_q T^p = [p]_q \cdot T^{p-1} \cdot d_q T$

$$\rightsquigarrow \varphi\left(\frac{d_q \log T}{q-1}\right) = \frac{d_q \log T}{q-1}.$$

(2) $(K^\bullet)^{\varphi = 1} \simeq [\widehat{\bigoplus_{n \in \mathbb{Z}[\frac{1}{p}]}} L \cdot T^n \xrightarrow{q^n - 1} \widehat{\bigoplus_{n \in \mathbb{Z}[\frac{1}{p}]}} L \cdot T^n \cdot \frac{d_q \log T}{q-1}]^{\varphi=1}$

completed colim over φ ($\varphi(d) = d^p$).

$$\simeq (R\Gamma_{pro\acute{e}t}(X_\eta, \mathcal{O}^\flat))^{\varphi=1} = R\Gamma_{\acute{e}t}(X_\eta, \mathbb{F}_p)$$

almost purity. Artin-Schreier seq.

Next: Derived prismatic cohom.

Def'n (A, I) bdd prism.

$\mathbb{\Delta}_{/A}$: {derived p-complete animated A/I} $\longrightarrow \widehat{D}(A)$

as left Kan ext'n of $R\Gamma_\Delta(-/A)$ on p-complete sm A/I-alg.

Then we obtain properties:

(1) left Kan extending $\tau^{\leq n} R\Gamma_\Delta(-/A)$ yields an increasing exhaustive $\mathbb{N}$-indexed fil'n $\mathrm{Fil}_n^{conj}(\Delta_{R/A})$ with

$$\mathrm{gr}_n^{conj}(\Delta_{R/A}) \simeq \Lambda^n(\widehat{\mathbb{L}}_{R/(A/I)})\{-n\}[-n].$$

(2) $R \longmapsto \Delta_{R/A}$ satisfies quasi-syntomic descent.

Def'n R p-complete with bdd p^∞-torsion.

(1) R <u>quasi-syntonic</u> if $\hat{\mathbb{L}}_{R/\mathbb{Z}_p}$ of p-complete tor-amplitude in $[-1,0]$

(2) $R \to R'$ morph <u>quasi-syntonic</u> if

$\hat{\mathbb{L}}_{R/R'}$ of p-complete tor-amplitude in $[-1,0]$

& $R \to R'$ is p-completely flat.

<u>Lemma</u> If $R \to R'$ quasi-syntonic cover with Čech nerve $R'^{\bullet}$

& $A/I \to R$ quasi-syn.

then $\Delta_{R/A} \simeq \varprojlim_{\Delta} \Delta_{R'^{\bullet}/A}$.

<u>Powerful strategy to access $\Delta_{-/A}$.</u>

(1) Reduce to Sm A/I-alg using left Kan-ext'n.

(2) Reduce to large A/I-alg R via q-syn descent.

$A/I \to R$ q-syn & exists surj $A/I\langle x_j^{1/p^\infty} \mid j \in J\rangle \twoheadrightarrow R$.

($\rightsquigarrow$ $\hat{\mathbb{L}}_{R/(A/I)}[-1]$ p-completely flat mod $/R$.)

<u>Consequence</u>: $\Delta_{R/A}$ is concentrated in deg 0

& initial in $(R/A)_\Delta$.

Sample application: Nygaard filtration.

<u>Theorem</u> R A/I-alg.

$\rightsquigarrow$ $\exists$ natural fil'n Fil^i_N on $\Delta^{(1)}_{R/A} := \Delta_{R/A} \otimes_{A,\varphi} A$ with

$\mathrm{gr}^i_N \Delta^{(1)}_{R/A} \simeq \mathrm{Fil}^{\mathrm{conj}}_i(\bar{\Delta}_{R/A})\{i\}$.

If R Sm $/(A/I)$, then

décalage w.r.t. I.

$\varphi_{R/A} : \Delta^{(1)}_{R/A} \xrightarrow{\sim} L\eta_I \Delta_{R/A}$.

<u>Sketch</u> If R large, set $\mathrm{Fil}^i_N \Delta^{(1)}_{R/A} := \varphi^{-1}(I^i \cdot \Delta_{R/A})$.

In particular, get de Rham comparison.

When R sm $/ (A/I)$,

$$\Delta_{R/A} \otimes_{A,\varphi} A \simeq (L\eta_I \Delta_{R/A}) \otimes_A A/I$$
$$\xrightarrow{\sim} \mathcal{H}^*(\overline{\Delta}_{R/A})\{*\} \simeq \Omega^*_{R/(A/I)}.$$