Prismatic cohomology (3/4)

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§ Prismatization

Fix prime p.

<u>Question</u> What is the "spectrum of the $\underbrace{\text{initial prism}}_{\text{(doesn't exist)}}$"?

<u>Comments</u> Set $A^0 := \mathbb{Z}_p\{x, \delta(x)^{-1}\}^{\wedge}_{(p,x)}$ = initial oriented prism.
$\cup\!\!\mid$
$I^0 = (x)$

Expect flat surjection $\mathrm{Spf}\, A^0 \longrightarrow Z$.

Also expect $\mathrm{Spf}\, A^0 \times_Z \mathrm{Spf}\, A^0 \simeq \mathrm{Spf}\, A^0 \times \mathrm{Spf}(\mathbb{Z}_p\{u^{\pm 1}\}^{\wedge}_p)$.

<u>Lemma</u> (1) As functors on rings, (free δ-ring)

$$\mathrm{Spec}\, \mathbb{Z}\{y\} = W, \quad \mathrm{Spec}\, \mathbb{Z}\{u^{\pm 1}\} = W^{\times}.$$

(2) As functors on $\mathrm{Nilp} := \{R \text{ } p\text{-nilpotent}\}$,

$$(\mathrm{Spf}\, A^0)(R) = \left\{ \sum_{i=0}^{\infty} V^i[a_i] : a_0 \in R \text{ nilp}, a_1 \in R^{\times} \right\}.$$

<u>Proof</u> (1) R ring.

$$\mathrm{Hom}_{\text{rings}}(\mathbb{Z}\{y\}, R) \simeq \mathrm{Hom}_{\delta\text{-rings}}(\mathbb{Z}\{y\}, W(R))$$
$$\simeq \mathrm{Hom}_{\text{rings}}(\mathbb{Z}\{y\}, W(R)) = W(R).$$

The rest follows.

$$\left(\begin{array}{l} A\ \delta\text{-ring}. \quad \varphi(x) = x^p + p\delta(x) \\ \qquad \Rightarrow \varphi^n(x) = x^{p^n} + p\delta(x)^{p^{n-1}} + p^2\delta_2(x)^{p^{n-2}} + p^n\delta_n(x). \\ \Rightarrow A \longrightarrow W(A), \ a \longmapsto (a, \delta(a), \delta_2(a), \dots) \end{array} \right)$$

<u>Def'n</u> $\mathrm{WCart} := [\mathrm{Spf}\, A^0 / W^\times]$.

<u>Lemma</u> R adic ring $/ \mathbb{Z}_p$

$\rightsquigarrow \mathrm{WCart}(R) := \mathrm{Hom}(\mathrm{Spf}\, R, \mathrm{WCart})$

$$\simeq \left\{ (I \xrightarrow{\alpha} W(R)) \;\middle|\; \begin{array}{l} I \in \mathrm{Pic}\, W(R), \\ \text{locally } \alpha = [a_0] + V[a_1] + \cdots \\ \quad \text{with } a_0 \in R \text{ nilp}, a_1 \in R^\times \end{array} \right\}$$

= "Cartier-Witt divisors".

Given (A, I) prism $\rightsquigarrow$ $A \xrightarrow{\mathrm{id}} A$ lifts to morph of δ-rings

$\gamma : A \longrightarrow W(A)$, $a \longmapsto (a, \delta(a), \ldots)$.

<u>Def'n</u> $\rho_A : \mathrm{Spf}\, A \longrightarrow \mathrm{WCart}$ given by

$$(I \otimes_{A,\gamma} W(A) \longrightarrow W(A)).$$

<u>Note</u> If $(d) = I$, then $\gamma(d) = (d, \delta(d), \ldots)$.

<u>Prop'n</u> (1) Assume (A, I), (B, J) bdd, (A, I) transversal.

(C, K) their product in $(\mathbb{Z}_p)_\Delta$.

Then $\exists$ induced morphs

$$\begin{tikzcd} \mathrm{Spf}\, C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spf}\, B \arrow[d, "\rho_B"] \\ \mathrm{Spf}\, A \arrow[r, "\rho_A"] & \mathrm{WCart} \end{tikzcd} \quad \text{Cartesian.}$$

(2) If (A, I) transversal, $A \neq 0$, then

ρ_A is faithfully flat.

(3) $$D(\mathrm{WCart}) := \varprojlim_{\substack{\mathrm{Spec}\, R \to \mathrm{WCart} \\ R \in \mathrm{Nilp}}} D(R) \simeq \varprojlim_{(A, I)} \hat{D}(A)$$

$=$ "prismatic crystals on $(\mathbb{Z}_p)_{\Delta}$".

In fact, $R\Gamma(\mathrm{WCart}, \mathcal{O}) \simeq R\Gamma((\mathbb{Z}_p)_{\Delta}, \mathcal{O}_{\Delta})$

Proof (1) WLoG, reduce to the case

$$B = A = A^{\circ}.$$

Then $\mathrm{Spf}\, C \simeq \mathrm{Spf}\, A^{\circ} \times \mathrm{Spf}\,(\mathbb{Z}_p\{u^{\pm 1}\}^{\wedge}_{p})$.

$$\simeq \mathrm{Spf}\, A^{\circ} \times_{\mathrm{WCart}} \mathrm{Spf}\, A^{\circ}.$$

(2) Consequence of the key tech lemma from lecture (2/4) on prismatic envelope.

(3) Follows from flat descent.

In particular, $R\Gamma((\mathbb{Z}_p)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \simeq R\Gamma(\mathrm{WCart}^{\mathrm{HT}}, \mathcal{O})$

$\mathrm{WCart}^{\mathrm{HT}} \subseteq \mathrm{WCart}$ locus where $a_0 = 0$.

$\mathrm{WCart}^{\mathrm{HT}} \times_{\mathrm{WCart}, \rho_A} \mathrm{Spf}\, A \simeq \mathrm{Spf}\, A/I$.

Lemma $\eta : \mathrm{Spf}\, \mathbb{Z}_p \xrightarrow{V(1)} \mathrm{WCart}^{\mathrm{HT}}$ is faithfully flat.

Proof Take $\alpha = V(u)$, $u \in W^{\times}(R)$.

Note $V(u) \cdot x = V(F(x) \cdot u)$.

$\rightsquigarrow$ Now, $F : W^{\times} \longrightarrow W^{\times}$ faithfully flat. $\square$

Lemma (1) $\underline{\mathrm{Aut}}\, \eta = W^{\times}[F] = \{x \in W^{\times} \mid F(x) = 1\}$.

(2) $\ker(W \xrightarrow{V(1)} W) = W[F] = \{x \in W \mid F(x) = 0\}$.

Proof $x \cdot V(1) = V(F(x) \cdot 1)$. $\square$

Prop'n (1) $W^{\times}[F] \simeq \mathbb{G}_m^{\sharp} = \mathrm{Spf}\left(\widehat{\bigoplus_{n \geq 0}} \mathbb{Z}_p \cdot \frac{(t-1)^n}{n!}\right)$

$=$ "PD-hull unit section of $\mathbb{G}_m$".

In particular, $\mathrm{WCart}^{HT} \simeq B\mathbb{G}_m^\sharp$

classifying stack for $\mathbb{G}_m^\sharp$.

(2) $W[F] \simeq \mathbb{G}_a^\sharp = \mathrm{Spf}\left(\widehat{\bigoplus_{n\geq 0}} \mathbb{Z}_p \cdot \frac{a^n}{n!}\right)$.

Proof (1) is slightly more complicated than (2).

(2)
$$\begin{tikzcd} W[F] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spf}\, \mathbb{Z}_p \arrow[d, "0"] \\ W \arrow[r, "F"] & W \end{tikzcd} \qquad \begin{tikzcd} W[F] \arrow[r] \arrow[d] & \mathbb{G}_a \\ \mathbb{G}_a^\sharp \arrow[ur] & \end{tikzcd}$$

$$\rightsquigarrow \quad W[F] \simeq \mathrm{Spf}\left(\underbrace{\mathbb{Z}_p\{y\}^\wedge}_{} \otimes_{\underbrace{\mathbb{Z}_p\{x\}}_{}} \mathbb{Z}_p\right)$$

$$\varphi(y) \longleftarrow x \longmapsto 0, \ \delta(x) \longmapsto 0$$

$$\simeq \mathrm{Spf}\left(\mathbb{Z}_p[y_0, y_1, \dots]_p^\wedge / (y_0^p + py_1, y_1^p + py_2, \dots)\right)$$

$$\simeq \mathbb{G}_a^\sharp. \qquad \square$$

Prop'n (1) $D(B\mathbb{G}_m^\sharp) = \left\{ \begin{array}{l} (M \in \hat{D}(\mathbb{Z}_p),\ \Theta : M \to M) \\ \text{where } \Theta^p - \Theta \text{ loc nilp on } H^*(M \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \end{array} \right\}$

(2) $D(B\mathbb{G}_a^\sharp) = \left\{ \begin{array}{l} (M \in \hat{D}(\mathbb{Z}_p),\ \Theta : M \to M) \\ \text{where } \Theta \text{ loc nilp on } H^*(M \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \end{array} \right\}$

Proof (Only (2) For vector bundles)

Set $A := \mathcal{O}_{\mathbb{G}_a^\sharp}$ $\mathbb{Z}_p$-Hopf-alg

$B = \mathrm{Hom}_{\mathbb{Z}_p}(A, \mathbb{Z}_p) \simeq \mathbb{Z}_p[\![\Theta]\!]$ is a power series alg.

If M fin free $\mathbb{Z}_p$-mod, then equiv

- a Coaction $M \to M \otimes_{\mathbb{Z}_p} A \simeq \mathrm{Homcont}(B, M)$
- an action $B \otimes_{\mathbb{Z}_p} M \longrightarrow M$ s.t. Θ is top nilp on M. $\square$

Rmk $0 \to \mu_p \to \mathbb{G}_m^\sharp \xrightarrow{\log} \mathbb{G}_a^\sharp \to 0$ ses of gp schs.

Corollary $R\Gamma((\mathbb{Z}_p)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) = R\Gamma(\mathrm{WCart}^{HT}, \mathcal{O})$

$$\simeq R\Gamma(B\mathbb{G}_m^\sharp, \mathbb{Z}_p) = (\mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p).$$

R p-complete. $(R \to B/J \leftarrow B) \in (R)_{\Delta}$.

Let $\gamma : B \to W(B)$ δ-map lifting Id_B.

$\rightsquigarrow$ get $R \to B/J \to B/J \otimes^{\mathbb{L}}_{B,\gamma} W(B)$

$$= \mathrm{Cone}(J \otimes_B W(B) \to W(B))$$

morph of animated rings.

Def'n (prismatization)

(1) $R^{\Delta} = \mathrm{WCart}_R :=$ stack on Nilp s.t.

$$R^{\Delta}(S) = \{(\underset{\mathrm{WCart}(S)}{I \xrightarrow{\alpha} W(S)}, \underset{\text{morph of animated rings}}{R \to \mathrm{Cone}(\alpha)})\}$$

In particular, $\mathbb{Z}_p^{\Delta} = \mathrm{WCart}$.

(2) (A, I) prism. R A/I-alg. Then

$$\begin{array}{ccc} (R/A)^{\Delta} := \mathrm{WCart}_{R/A} & \longrightarrow & \mathrm{WCart}_R = R^{\Delta} \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, A & \xrightarrow{\rho_A} & \mathrm{WCart} =: \mathbb{Z}_p^{\Delta}. \end{array}$$

Thm (Bhatt-Scholze, generalized ver.)

(1) R quasi-regular semiperf'd.

(R q-Syn, $\exists R_0 \twoheadrightarrow R$, R_0 perf'd).

$\Rightarrow \quad R^{\mathbb{\Delta}} \simeq \mathrm{Spf}(\mathbb{\Delta}_R)$

$\qquad\qquad\qquad \mathbb{\Delta}_R = \mathbb{\Delta}_{R/A_{inf}(R_0)}$.

(2) R quasi-syntomic. Then

$$D(R^{\mathbb{\Delta}}) \simeq \{\text{prismatic crystals on } (R)_{\mathbb{\Delta}}\}$$

In particular, $R\Gamma((R)^{\mathbb{\Delta}}, \mathcal{O}) \simeq R\Gamma((R)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$.

(3) (A, I) bdd prism, $A/I \to R$ quasi-syntomic.

$\rightsquigarrow D((R/A)^{\mathbb{\Delta}}) \simeq \{\text{prismatic crystals on } (R/A)_{\mathbb{\Delta}}\}$.

<u>Why is prismatization useful?</u>

Have a natural ring stack $\mathcal{R}$ over $W\mathrm{Cart} = \mathbb{Z}_p^{\mathbb{\Delta}}$

with $\mathcal{R}(S) := \mathrm{Cone}(\alpha)$, for $(I \xrightarrow{\alpha} W(S)) \in W\mathrm{Cart}(S)$.

In fact, $\mathcal{R} = (\mathbb{Z}_p\langle x\rangle)^{\mathbb{\Delta}}$.

Given R p-complete, then

$$R^{\mathbb{\Delta}}(S) = \mathrm{Hom}_{\text{ring gpoids}}(R, \mathcal{R}(S)), \quad S \in \mathrm{Nilp}_{W\mathrm{Cart}}.$$

Set $\mathcal{R}^{dR} = \mathrm{Spf}\,\mathbb{Z}_p \times_{W\mathrm{Cart}} \mathcal{R}$ with $(\mathrm{Spf}\,\mathbb{Z}_p \xrightarrow{p} W\mathrm{Cart})$

$\qquad\qquad W(\mathbb{Z}_p) \to W(\mathbb{F}_p)$, $p = V(1)$.

<u>Lemma</u> $\mathcal{R}^{dR} \simeq [\mathbb{G}_a/\mathbb{G}_a^{\#}]$ over $\mathrm{Spf}\,\mathbb{Z}_p$.

<u>Proof</u> $\mathbb{G}_a/\mathbb{G}_a^{\#} \xleftarrow{\sim} W/VW \oplus \mathbb{G}_a^{\#} \xrightarrow[F]{\sim} W/FVW \simeq W/p = \mathcal{R}^{dR}$.

$(\mathbb{G}_a \simeq W/VW)$ $\quad F: W \to W$ faithfully flat

$\qquad \& \ \mathbb{G}_a^{\#} = W[F]$.

∎

<u>Cor</u> k perfect field, $k \to R$ sm.

Then $\varphi^*_{W(k)} R\Gamma_{\mathbb{\Delta}}(R/W(k)) \simeq R\Gamma_{crys}(R/W(k))$.

Proof $R\Gamma_{\Delta}(R/W(k)) \simeq R\Gamma((R/W(k))^{\Delta}, \mathcal{O})$ $\leftarrow$ b/c

$$\begin{tikzcd} \mathrm{Spf}\, W(k) \arrow[r] \arrow[dr] & \mathrm{WCart} \\ & \mathrm{Spf}\, \mathbb{Z}_p \arrow[u, "p"'] \end{tikzcd}$$

$$\simeq R\Gamma(\underbrace{\mathcal{H}om(R, \mathcal{R}^{dR}_{W(k)})}_{\text{stack on } \mathrm{Nilp}_{W(k)}})$$

$$\simeq R\Gamma(\mathcal{H}om(R, [\mathbb{G}_a/\mathbb{G}_a^{\#}]), \mathcal{O}).$$

If $\tilde{R}$ is a smooth lift of R to $W(k)$, then

$$\mathrm{Spf}\, \tilde{R} \xrightarrow[\text{surj}]{\text{flat}} \mathcal{H}om(R, \mathbb{G}_a/\mathbb{G}_a^{\#})$$

with Cech nerve given by PD-envelopes of the diagonals.

Thus, $\varphi^*_{W(k)} R\Gamma_{\Delta}(R/W(k)) \simeq R\Gamma_{crys}(R/W(k))$. $\square$

<u>Another application</u>:

(A, I) bdd prism. Set $\mathcal{R}^{HT} := \mathrm{Spf}\, A/I \times_{\mathrm{WCart}} \mathcal{R} \simeq \mathrm{Spf}\, A/I \times_{\mathrm{WCart}^{HT}} \mathcal{R}_{\mathrm{WCart}^{HT}}$.

<u>Lemma</u> $\exists$ natural morph of ring stacks

$$\mathcal{R}^{HT} \longrightarrow \mathbb{G}_a$$

whose kernel is $\underbrace{\mathbb{G}_a^{\#}\{1\}[1]}_{\text{square zero ideal}}$.

($M \to C \to C_0$ of animated rings

$\rightsquigarrow$ square zero ext'n

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C_0 \arrow[d, "(id, 0)"] \\ C_0 \arrow[r, "(id, \delta)"] & C_0 \oplus M[1]. \end{tikzcd}$$
)

Proof Consider $\pi_0(\mathcal{R}^{HT}(S)) = W(S)/V(1) \cdot W(S)$

$$\longrightarrow S = W(S)/VW(S).$$

$(V(1) \cdot W(S) = V(F(W(S))).)$

$$\pi_1(\mathcal{R}^{HT}(S)) \simeq I/I^2 \otimes_{A/I} W[F](S) \simeq \mathbb{G}_a^{\#}\{1\}(S).$$ $\square$

Cor Assume $R \to A/I$ smooth. Set $\bar{A} = A/I$.

Then $(R/A)^{HT} \longrightarrow \mathrm{Spf}\, R$ is a gerbe bounded by

$J^{\#}_{R/\bar{A}}\{1\}$ = "PD-hull of zero section in $J_{R/\bar{A}}\{1\}$"

$$\overset{\shortparallel}{\mathrm{Spf}(\mathrm{Sym}^{\bullet}_R(\hat{\Omega}^1_{R/\bar{A}}\{1\}))}.$$

Proof By derived deformation theory.

Cor $H^i((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \simeq \hat{\Omega}^i_{R/\bar{A}}\{-i\}$.

(Calculated by $[0 \xrightarrow{0} \hat{\Omega}_{R/\bar{A}}\{-1\} \xrightarrow{0} \cdots]$

& $R\Gamma(B\mathbb{G}_a^{\#}, M) = (M \xrightarrow{\Theta} M)$.)