Prismatic cohomology (4/4)

Johannes Anschütz

§ The prismatic logarithm

(A, I) any prism.

Aims (1) Construct a natural invertible A-mod $A\{1\}$ (Breuil-Kisin twist) with the natural data:

- an isom $\varphi_{A\{1\}} : \varphi^* A\{1\} \xrightarrow{\sim} I^{-1} \cdot A\{1\}$

  Write $\varphi_{A\{1\}} : A\{1\} \xrightarrow{\text{can}} \varphi^* A\{1\} \xrightarrow{\varphi_{A\{1\}}} A\{1\}$.

- an isom $A\{1\} \otimes_A A/I \simeq I/I^2$.
- if $I = (p)$ then $\exists$ a trivialization $A \simeq A\{1\}$

Heuristic $A\{1\}$ " = " $I \otimes_A \varphi^* I \otimes_A \cdots$

(2) Construct a natural homomorphism

$$\log_{\Delta} : (1+I)^{\mathrm{rk}=1} \longrightarrow A\{1\}.$$
$$(1+I)^{\mathrm{rk}=1} = \{x \in 1+I \mid \delta(x) = 0\}$$

s.t.
- $\varphi_{A\{1\}}(\log_{\Delta}(x)) = \log_{\Delta}(x)$
- $\log_{\Delta}(x) \equiv x - 1 \in I/I^2$.

Consequences:
- Breuil-Kisin twists

$$M\{j\} := M \otimes_A A\{j\}, \quad j \in \mathbb{Z}.$$

- Set $\mathcal{N}^{\geq \bullet} A := \varphi^{-1}(I^{\bullet})$ Nygaard fil'n.

$$\rightsquigarrow \quad \varphi_j : \mathcal{N}^{\geq j} A\{j\} \longrightarrow A\{j\}$$
$$a \otimes b \longmapsto \varphi(a) \cdot \varphi_{A\{j\}}(b).$$

Concretely, assume $I = (d)$. $A\{1\} = A \cdot e$, $\varphi_{A\{1\}}(e) = \frac{1}{d} \cdot e$.

$$\rightsquigarrow \quad \mathcal{N}^{\geq j}A \overset{e^j}{\simeq} \mathcal{N}^{\geq j}A\{j\} \xrightarrow{\varphi_j} A\{j\} \overset{e_j}{\simeq} A$$
$$a \longmapsto \frac{\varphi(a)}{d^j}, \quad j \in \mathbb{Z}.$$

Note $\log_\Delta(-) \in \mathcal{N}^{\geq 1}A\{1\}$.

Set $R := A/\mathcal{N}^{\geq 1}A$. $\rightsquigarrow$ get $R \longrightarrow A/I$
$$\bar{a} \longmapsto \overline{\varphi(a)}.$$

Set $[-] : R^\flat \longrightarrow (A/I)^\flat \longrightarrow A$ Teichmüller lift
$$R^\flat = \varprojlim_{x \mapsto x^p} R$$

$\rightsquigarrow [T_p R^\times]$ (p-adic Tate mod)
$$T_p R^\times = \{(1, r_1, \dots) \in R^\flat\}.$$

Get
$$T_p R^\times \xrightarrow{\log_\Delta([-])} \mathrm{eq}\left(\mathcal{N}^{\geq 1}A\{1\} \underset{\mathrm{can}}{\overset{\varphi_1}{\rightrightarrows}} A\{1\}\right)$$
$$z \longmapsto [z] - 1 \in A\{1\} \otimes_A A/I \simeq I/I^2.$$

If (A, I) transversal, then the downstair map is actually injective. (will show later.)

Example Assume $(\mathbb{Z}_p[[q-1]], ([p]_q)) \longrightarrow (A, I)$
$$\rightsquigarrow q^p \in (1+I)^{\mathrm{rk}=1}.$$

Lemma (later) $A\{1\} = A \cdot e_A$ with $(q-1)\cdot e_A = \log_\Delta(q^p)$.
& $\varphi_{A\{1\}}(e_A) = [p]_q^{-1} \cdot e_A$.

Def'n
$$\log_q(u) := \sum_{n=1}^{\infty} (-1)^{n-1} \cdot q^{-n(n-1)/2} \cdot \frac{(u-1)(u-q)\cdots(u-q^{n-1})}{[n]_q}.$$

Have $\log_q(u) \equiv \log(u) \mod q-1$, and $\log_q(n)$ "=" $\dfrac{(q-1)}{\log q}\cdot \log n$.

(only holds when $\log q$ makes sense.)

Unique series s.t. $\log_q(1) = 0$ & $\nabla_q(\log_q(u)) = \frac{1}{u}$,

i.e. $\dfrac{\log_q(qu) - \log_q u}{q-1} = 1$.

<u>Lemma</u> $\log_\Delta(u^p) = \log_q(u)\cdot e_A$ for $u \in A^{rk=1}$ s.t. $\varphi(u) \in 1 + I$.

<u>Proof</u> Write $\log_\Delta(u^p) = f(u)\cdot e_A$.

Check: $f(1) = 0$.

$\log_\Delta(q^p \cdot u^p) - \log_\Delta(u^p) = (q-1)\cdot e_A$. □

<u>Note</u> let $\nu(-) : R^\flat \longrightarrow A$ Teichmüller lift for the surj $A \twoheadrightarrow R$.

$\rightsquigarrow$ $\varphi(\nu(-)) = [-]$ b/c
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ R & \longrightarrow & A/I \end{array}.$$

If $x \in T_pR^\times$, for $\log_\Delta([x]) = \log_\Delta(\varphi(\nu(x)))$
$= \log_\Delta(\nu(x)^p) = \log_q(\nu(x))\cdot e_A$

Thus, if R quasi-regular semi-perfect (qrsp) $\mathbb{F}_p$-alg, $A = \Delta_R$,

then $\log_\Delta([x]) = \log_q(\nu(x)) \in \Delta_R^{\varphi = [p]_q} \simeq \Delta_R^{\varphi = p}$

$\log q/(q-1)$.

$\log(\nu(x))$.

$q \in \mathbb{Z}_p[[q-1]] \longmapsto [\varepsilon]$

If $R = \mathcal{O}_C/p \Rightarrow A_{inf}(\widehat{\mathcal{O}_C}) \longrightarrow \Delta_R$.

§ Construction of AFiS.

(I) Following Drinfeld.

Recall: $\{$line bundles for $\mathcal{O}_{\Delta}$ on $(\mathbb{Z}_p)_{\Delta}\}$
$\simeq \{$line bundles on $W\mathrm{Cart}\}$.

Note: Have $F: W\mathrm{Cart} \to W\mathrm{Cart}$ induced $F: W \to W$.

Want: $\mathcal{O}\{i\} \in \mathrm{Pic}(W\mathrm{Cart})$ + isom
$F^*\mathcal{O}\{i\} \simeq \mathcal{I}^{-1}\mathcal{O}\{i\}$ with $\mathcal{I}$ = ideal sheaf of $W\mathrm{Cart}^{HT}$.

Lem $\mathbb{Z}_p \simeq H^0(W\mathrm{Cart}, \mathcal{O})$ $\quad (\mathbb{Z}_p[\![q-1]\!]^{\mathbb{Z}_p^\times} = \mathbb{Z}_p)$.

Set $\rho := \rho_{\mathbb{Z}_p} : \mathrm{Spf}\,\mathbb{Z}_p \longrightarrow W\mathrm{Cart}$.

Cor $\mathrm{Pic}^1 = \{\mathcal{L}$ line bundle on $W\mathrm{Cart}$ & trivialization $\mathbb{Z}_p \simeq \rho^*\mathcal{L}\}$
is equiv to a set.

E.g. $\mathcal{I}$ naturally defines an object using $\rho^*\mathcal{I} \cong (p) \xleftarrow[\sim]{p} \mathbb{Z}_p$.

Lem $\mathrm{Id} - F^* : \mathrm{Pic}^1 \xrightarrow{\sim} \mathrm{Pic}^1$
$\mathcal{L} \longmapsto \mathcal{L} \otimes F^*\mathcal{L}^{-1}$.

Thus, $\mathcal{O}\{1\} := (\mathrm{Id} - F^*)^{-1}(\mathcal{I})$.

Proof Given $\mathcal{I}$ set $\bigotimes_{i=0}^{\infty} (F^i)^*\mathcal{L}$.

Well-def'd: Given $S \in \mathrm{Nilp}$, $\mathrm{Spec}\, S \to W\mathrm{Cart}$
$\rightsquigarrow \exists\, i_0 \geq 0$ s.t. $i \geq i_0$,

$$\begin{tikzcd} \mathrm{Spec}\, S \arrow[r] \arrow[d, dashed] & W\mathrm{Cart} \arrow[d, "F^i"] \\ \mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "\rho"] & W\mathrm{Cart} \end{tikzcd}$$

factors canonically over $\rho: \mathrm{Spf}\,\mathbb{Z}_p \longrightarrow W\mathrm{Cart}$.

Use (i) Given $\alpha = [a_0] + Vb$ with $a_0^{i} = 0$.
$\rightsquigarrow F^i(\alpha) = p \cdot \underbrace{F^{i-1}b}_{\in W^*(S)}$.

(ii) Exists $i \geq 0$, s.t. $\forall u \in W^\times(S)$ with $pu = p$,

then $F^i(u) = 1$. $\square$

<u>Lemma</u> $\mathcal{O}\{1\}|_{WCart^{HT}} \simeq I|_{WCart^{HT}}$.

<u>Proof</u> STS:

$$\begin{tikzcd} WCart^{HT} \arrow[r] \arrow[d, "\exists\, \mathrm{isom}"'] & WCart \arrow[d, "F"] \\ \mathrm{Spf}\, \mathbb{Z}_p \arrow[r, "p"] & WCart. \end{tikzcd}$$

Indeed, $FV(u) = p \cdot u$

$\underline{\mathrm{Aut}}(W(S) \xrightarrow{V(u)} W(S)) = W^\times[F]$. $\square$

(II) Following Bhatt–Lurie.

WLOG, (A, I) transversal.

Set $I_r := I \cdot \varphi^* I \cdots (\varphi^*)^{r-1} I$. Let $\pi_r : I_r/I_r^2 \to I_{r-1}/I_{r-1}^2$.

If $I = (d)$ then $\varphi(d)$ non-zero divisor.

<u>Lemma</u> (1) $I_r \subseteq A$ invertible

(2) $A \xrightarrow{\sim} \varprojlim_r A/I_r$, $A/I_r \hookrightarrow \prod_{0 \leq s < r} A/(\varphi^s)^* I$

($\rightsquigarrow$ Consequence: $A\{1\}^{\varphi_{A\{1\}} = 1} \hookrightarrow I/I^2$.)

(3) π_r is divisible by p & $\frac{1}{p} \cdot \pi_r : I_r/I_r^2 \twoheadrightarrow I_{r-1}/I_{r-1}^2$ surj.

- Set $A\{1\} := \varprojlim_r (\cdots \to I_r/I_r^2 \xrightarrow{\frac{1}{p}\pi_r} I_{r-1}/I_{r-1}^2 \to \cdots)$

Breuil–Kisin twist

an invertible A-mod.

(4) $A\{1\} \otimes_A A/I \simeq I/I^2$ via first proj

(5) $A\{1\} \otimes_A A\{\frac{d}{p}\}_p^\wedge$ is canonically trivial, $I = (d)$.

<u>Proof</u> (Sketch) (1) WLOG, assume $I = (d)$ & (p, d) regular.

$\rightsquigarrow$ $(p, \varphi^r(d))$ also regular

$\Rightarrow$ $(\varphi^r(d), p)$ reg $\Rightarrow$ $\varphi^r(d)$ injective.

(2) Check $A \simeq \varprojlim A/I_r$ mod p.

For rest $r=2$ for simplicity. $I=(d)$.

Take $\eta = \varphi(d)\cdot x \Rightarrow \varphi(d)\cdot x \equiv p\cdot\delta(d)\cdot x \mod I$.

If $\eta \in I$ then $d \mid x \Rightarrow A/I_2 \hookrightarrow A/I \times A/\varphi^* I$.

Now $I_2 \longrightarrow I/I_2$, $d\cdot\varphi(d) \longmapsto d(d^p + p\delta(d)) \equiv p\cdot\delta(d)$.

(4) Follows from the pf above (with $I=(p)$, $d = p\cdot\frac{d}{p}$.)

(5) $I_r \cdot A\{\frac{d}{p}\}_p^\wedge = (p^r) \xrightarrow{1/p} (p^{r-1}) = I_{r-1} A\{\frac{d}{p}\}_p^\wedge / I_{r-1}$. $\square$

Set $\varphi_{A\{1\}} : \varphi^* A\{1\} \xrightarrow{\sim} I^{-1} A\{1\}$ is induced by $\varphi^* I_{r-1} \simeq I^{-1} \cdot I_r$.

More precisely, $\varphi^* I_{r-1} \otimes_A A/\varphi^* I_{r-1} \simeq I^{-1}\cdot I_r \otimes_A A/\varphi^* I_{r-1}$

$$\begin{array}{ccc} \varphi^* I_{r-1} \otimes_A A/\varphi^* I_{r-1} & \simeq & I^{-1}\cdot I_r \otimes_A A/\varphi^* I_{r-1} \\ \text{is} & & \text{is} \\ \varphi^*(I_{r-1}/I_{r-1}^2) & & I^{-1}\cdot I_r \otimes_A A/I_r \otimes_{A/I_r} A/\varphi^* I_{r-1}. \\ & & \text{is} \\ & & I^{-1}\cdot A\{1\} \otimes_A A/\varphi^* I_{r-1}. \end{array}$$

$\rightsquigarrow$ pass to inverse lim over r.

Next prismatic logarithm

WLOG. (A, I) transversal.

Prop'n Let $u \in (1+I)^{rk=1}$. Then

(1) $u^{p^r} - 1 \in I_{r+1}$.

(2) $\frac{1}{p}\pi_r : I_{r+1}/I_{r+1}^2 \longrightarrow I_r/I_r^2$ maps $u^{p^r}-1$ to $u^{p^{r-1}}-1$.

$\rightsquigarrow$ Set $\log_\Delta(u) := (\underset{I/I^2}{\underset{\cap}{u-1}}, \underset{I_2/I_2^2}{\underset{\cap}{u^p-1}}, \dots) \in A\{1\}$.

(3) $\log_\Delta(-)$ is a group homomorph.

(4) $\varphi_{A\{1\}}(\log_{\Delta}(-)) = \log_{\Delta}(-)$.

Proof (1) Use $A/I_{r+1} \hookrightarrow \prod_{0\le s\le r} A/(\varphi^s)^* I$.

$$\& \quad \underset{\displaystyle u^{p^r}-1}{\varphi^s}\!\!\!\!\!\!\overset{}{}\;\; (\underset{\displaystyle I}{\underset{\cap}{u^{p^{r-s}}}} - 1) \in (\varphi^s)^* I.$$

(2) $x = u^{p^{r-1}} \overset{(1)}{\Longrightarrow} x \equiv 1 \bmod I_r$.

$$\Rightarrow \underbrace{(x-1)(1+x+\cdots+x^{p-1})}_{\displaystyle \overset{\shortparallel}{x^p-1 = u^{p^r}-1}} \equiv \underbrace{p(x-1)}_{\displaystyle \overset{\shortparallel}{p(u^{p^{r-1}}-1)}} \bmod I_r^2 .$$

(4) $\varphi(u^{p^r}-1) = u^{p^{r+1}}-1$. $\square$

Assume $(\mathbb{Z}_p[[q-1]], ([p]_q)) \longrightarrow (A, I)$.

Lemma $A\{1\} = A\cdot e_A$ with $(q-1)\cdot e_A = \log_{\Delta}(q^p)$.

$$\varphi_{A\{1\}}(e_A) = [p]_q^{-1}\cdot e_A .$$

Proof $I_r = ([p^r]_q) = \left(\frac{q^{p^r}-1}{q-1}\right)$. Define e_A via $\frac{q^{p^r}-1}{q-1} \in I_r/I_r^2$. $\square$