Lecture C1 "Some arithmetic applications"

§0. Motivation

$E/\mathbb{Q}$ elliptic curve, cond $E = N$

$p \nmid 2N$ good <u>supersingular</u> prime : $a_p := p+1-\#E(\mathbb{F}_p) \equiv 0 \pmod p$.

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) \to Ш(E/\mathbb{Q})[p^\infty] \to 0$$

Consider the following (1)–(3) :

(1) <u>p-part of BSD formula in rk 0</u>

$$L(E,1) \neq 0 \implies \mathrm{ord}_p\left(\frac{L(E,1)}{\Omega_E}\right) = \mathrm{ord}_p\left(\# Ш(E/\mathbb{Q}) \prod_{\ell | N} c_\ell(E/\mathbb{Q})\right)$$

(2) <u>p-converse to Gross-Zagier & Kolyvagin</u>

$$\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \implies \mathrm{ord}_{s=1} L(E,s) = 1.$$

(3) <u>p-part of BSD formula in rk 1</u> :

$$\mathrm{ord}_{s=1} L(E,s) = 1$$

$$\Rightarrow \mathrm{ord}_p\left(\frac{L'(E,1)}{\Omega_E \cdot \mathrm{Reg}_E}\right) = \mathrm{ord}_p\left(\#Ш(E/\mathbb{Q}) \cdot \prod_{\ell | N} c_\ell(E/\mathbb{Q})\right).$$

<u>Today</u>: Assuming further $a_p = 0$ (automatic $p>3$),

explain how (1)–(3)

follow from certain "signed" Main Conjectures

pioneered by Kobayashi.

(Later lecture: How "Main result" in Xin's lecture

$\Downarrow$

these "signed" Main Conjectures).

From now on: $a_p = 0$.

§ 1. Kobayashi's Main Conj.

$$\bigcup_{n \geq 0} \mathbb{Q}_n = \mathbb{Q}_\infty \subset \mathbb{Q}(\mu_{p^\infty})$$

$$\begin{array}{c} \mathbb{Q}_\infty \\ \Big| \quad \Gamma \cong \mathbb{Z}_p \text{ cyclotomic } \mathbb{Z}_p\text{-ext'n.} \\ \mathbb{Q} \end{array}$$

Historical difficulty: The cokernel of restriction

$$\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n) \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^{\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}_n)}$$

is infinite for $n \gg 0$ (so Mazur's control thm. doesn't hold)

& the Pontryagin dual

$$X(E/\mathbb{Q}_\infty) := \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_\infty)^\wedge$$

is f.g. but non-torsion over $\Lambda := \mathbb{Z}_p[[\Gamma]] \; (\cong \mathbb{Z}_p[[X]]$

$$\gamma \longleftrightarrow 1+X$$

Another difficulty: The are 2 p-adic L-functions

(by Amice-Vélu & Vishik)

$$\mathcal{L}_{p,\alpha}(E),\ \mathcal{L}_{p,\bar{\alpha}}(E) \in \mathbb{Q}_p[\![\Gamma]\!] \supset \Lambda$$

where $\alpha, \bar{\alpha}$ = roots of $x^2 - a_p x + p$

$= \pm\sqrt{-p}$.

with the interpolation property: $\forall \chi : \Gamma \to \mu_{p^\infty}$

$$\mathcal{L}_{p,\alpha}(E)(\chi) = \begin{cases} \left(1-\frac{1}{\alpha}\right)^2 \cdot \frac{L(E,1)}{\Omega_E} & \text{if } \chi = \mathbb{1} \\ \frac{p^n}{\tau(\bar{\chi})\,\alpha^n} \cdot \frac{L(E,\bar{\chi},1)}{\Omega_E} & \text{if } \operatorname{cond}(\chi) = p^n > 1 \end{cases}$$

(similarly for $\mathcal{L}_{p,\bar{\alpha}}(E)$)

But $\operatorname{ord}_p(\alpha) > 0 \Rightarrow \mathcal{L}_{p,\alpha}(E) \notin \Lambda$.

Solutions:

<u>Kobayashi</u>: Consider the signed groups

$$\mathrm{Sel}^{\pm}_{p^\infty}(E/\mathbb{Q}_n) := \ker\left[\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_n) \xrightarrow{\mathrm{res}_p} \frac{E(\mathbb{Q}_{p,n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}{E^{\pm}(\mathbb{Q}_{p,n})}\right],$$

where

$$E^{+}(\mathbb{Q}_{p,n}) := \left\{ P \in E(\mathbb{Q}_{p,n}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \ \text{ s.t. } \ \mathrm{Tr}_{n/m+1}(P) \in E(\mathbb{Q}_{p,m}) \ \text{ for all } 0 \leq m < n, \ m \text{ even} \right\}$$

$$E^{-}(\mathbb{Q}_{p,n}) = \left\{ \text{——————————————} \ m \text{ odd} \right\}.$$

<u>Pollack</u>: $\exists\, \mathcal{L}_p^{+}(E), \mathcal{L}_p^{-}(E) \in \Lambda$

s.t.
$$\mathcal{L}_{p,\alpha}(E) = \mathcal{L}_p^{-}(E)\cdot \log_p^{+} + \mathcal{L}_p^{+}(E)\cdot \log_p^{-}\cdot \alpha$$

$$\mathcal{L}_{p,\bar{\alpha}}(E) = \text{——————————————} \ \bar{\alpha},$$

where $\log_p^{\pm} \in \mathbb{Q}_p[\![\Gamma]\!]$ are "half logarithms".

Kobayashi's Main Conjecture:

$$X^{\pm}(E/\mathbb{Q}_\infty) := \left(\varinjlim_n \mathrm{Sel}^{\pm}_{p^\infty}(E/\mathbb{Q}_n)\right)^{\wedge} \text{ is } \Lambda\text{-torsion},$$

with

$$\mathrm{char}_\Lambda X^{\pm}(E/\mathbb{Q}_\infty) = \left(\mathcal{L}_p^{\pm}(E)\right).$$

Proposition 1. Kobayashi's MC $\Rightarrow$ p-part of BSD formula in rk 0.

Proof. By Kobayashi's analogue of Mazur's control thm., the restriction map

$$\underbrace{\mathrm{Sel}^{+}_{p^\infty}(E/\mathbb{Q})}_{=\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})} \hookrightarrow \mathrm{Sel}^{+}_{p^\infty}(E/\mathbb{Q}_\infty)^{\Gamma}$$

is injective with finite cokernel.

Thus

$$L(E,1) \neq 0 \overset{\text{interpolation property}}{\Longrightarrow} \mathcal{L}_p^+(E)(0) \neq 0$$

$$\underset{\substack{\text{Kobayashi's} \\ \text{MC}}}{\Longrightarrow} \# X^+(E/\mathbb{Q}_\infty)_\Gamma < \infty$$

$$\& \ \underbrace{\# \mathrm{Sel}_{p^\infty}(E/\mathbb{Q})}_{= \# Ш(E/\mathbb{Q})[p^\infty]} < \infty.$$

Let $\mathcal{F}^+(E/\mathbb{Q}_\infty) \in \Lambda$ a char. power series for $X^+(E/\mathbb{Q}_\infty)$.

Then

$$\underbrace{\mathcal{F}^+(E/\mathbb{Q}_\infty)(0)}_{\substack{\| \bmod \mathbb{Z}_p^\times \\ \mathcal{L}_p^+(E)(0) = 2 \cdot \frac{L(E,1)}{\Omega_E}.}} \sim_p \# Ш(E/\mathbb{Q})[p^\infty] \prod_{\ell \mid N} c_\ell(E/\mathbb{Q})$$

$\therefore$ p-part of BSD formula holds. ▨

§2. Signed Heegner Point Main Conj.

$K/\mathbb{Q}$ imaginary quadr. field satisfying
the Heegner hyp.:

$\exists$ integral ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$

& such that

$$p = \mathfrak{p}\bar{\mathfrak{p}} \quad \text{splits in } K.$$

$$\begin{tikzcd}
K_\infty^{ac} = \bigcup_{n\geq 0} K_n^{ac} \arrow[d, dash, "\Gamma^{ac} \cong \mathbb{Z}_p \text{ anti-cyclotomic } \mathbb{Z}_p\text{-ext}^n"] \arrow[dd, dash, bend right=60, "\Gamma^{ac} \rtimes \langle\tau\rangle"'] \\
K \arrow[d, dash, "\langle\tau\rangle"] \\
\mathbb{Q}
\end{tikzcd}$$

Via $\pi_E : X_0(N) \to E$ get Heegner pts

$x_n \in E(K_n^{ac})$ s.t.

$$Tr_{n+1/n}(x_{n+1}) = \underbrace{a_p}_{\stackrel{\shortparallel}{0}} x_n - x_{n-1} \quad \forall n > 1$$

$\Rightarrow$ unbounded classes

$$\kappa_\infty^\alpha, \kappa_\infty^{\bar\alpha} \in \left(\varprojlim_n Sel(K_n^{ac}, T_pE) \right) \otimes_{\Lambda^{ac}} \mathbb{Q}_p[\![\Gamma^{ac}]\!],$$

where $\Lambda^{ac} := \mathbb{Z}_p[\![\Gamma^{ac}]\!]$.

C.-Wan: $\exists\ \kappa_\infty^+, \kappa_\infty^- \in \check{S}_p^{\pm}(K_\infty^{ac}, T_pE)$

$$\stackrel{\shortparallel}{\varprojlim_n} Sel^{\pm}(K_n^{ac}, T_pE)$$

defined following Kobayashi

s.t.
$$\begin{pmatrix} \kappa_\infty^{\alpha} \\ \kappa_\infty^{\bar\alpha} \end{pmatrix} = \underbrace{M_{\log}}_{\in M_{2\times 2}(\mathbb{Q}_p[[\Gamma^{ac}]])} \begin{pmatrix} \kappa_\infty^{+} \\ \kappa_\infty^{-} \end{pmatrix}$$

"half-logarithm matrix"

<u>Signed Heegner Point Main Conj.</u>:

$$X^{\pm}(E/K_\infty^{ac}) := \left(Sel_{p^\infty}^{\pm}(E/K_\infty^{ac})\right)^{\wedge}$$

has Λ^{ac}-rk 1, and

$$\operatorname{char}_{\Lambda^{ac}}\left(X^{\pm}(E/K_\infty^{ac})_{tors}\right) = \operatorname{char}_{\Lambda^{ac}}\left(\frac{\check{S}_p^{\pm}(K_\infty^{ac}, T_pE)}{(\kappa_\infty^{\pm})}\right)^2 \frac{1}{c_E^2 u_K^2}$$

where $u_K = \frac{1}{2}\#\mathcal{O}_K^{\times}$

c_E = Manin constant : $\pi_E^* \omega_E = c_E \cdot 2\pi i f(z)dz$

newform

Proposition 2. Signed Heegner Point MC

$\Longrightarrow$ p-converse to GZ & Kolyvagin.

Proof. Supp. $\operatorname{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$.

Choose $K/\mathbb{Q}$ imag. quadr. s.t.

- Heegner hyp. holds.
- $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K.
- $L(E^K, 1) \neq 0$.

By Kato,

$$\begin{array}{ccc} \operatorname{cork}_{\mathbb{Z}_p} \underbrace{\mathrm{Sel}_{p^\infty}(E/K)}_{\substack{\| \\ \mathrm{Sel}^+_{p^\infty}(E/K)}} & = \operatorname{cork}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1. \\ \| \; \text{control thm.} & \\ \mathrm{rk}_{\mathbb{Z}_p} X^+(E/K^{ac}_\infty)_{\Gamma^{ac}} & \end{array}$$

$$\overset{\text{signed HPMC}}{\Longrightarrow} (\gamma^{ac}-1) \nmid \mathrm{char}_{\Lambda^{ac}}\left(\frac{\check{S}_p^{+}(K_\infty^{ac}, T_pE)}{(\kappa_\infty^{+})} \right),$$

where $\gamma^{ac} \in \Gamma^{ac}$ top. gen.

$\Longrightarrow$ κ_∞^{+} has <u>non-torsion</u> image κ_0 under

$$\begin{tikzcd}
\check{S}_p^{+}(K_\infty^{ac}, T_pE)_{\Gamma^{ac}} \arrow[r, hook] & \mathrm{Sel}^{+}(K, T_pE) \arrow[d, phantom, "\shortparallel"] \\
\rotatebox{90}{$\in$} & \mathrm{Sel}(K, T_pE) \\
\kappa_\infty^{+} \arrow[r, mapsto] & \kappa_0 \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

But κ_0 = Kummer image of classical Heegner pt.

$$\overset{\text{Gross-Zagier}}{\Longrightarrow} \quad L'(E/K, 1) \neq 0$$

$$\overset{L(E^K,1)\neq 0}{\Longrightarrow} \quad \mathrm{ord}_{s=1} L(E,s) = 1 \qquad \blacksquare$$

§3. p-adic Gross-Zagier formula

Proposition 3 Kobayashi's MC

$\Rightarrow$ p-part of BSD formula in rk 1.

Proof. Suppose $\mathrm{ord}_{s=1} L(E,s) = 1$,

and choose $K/\mathbb{Q}$ imag. quadr. field

as in Proposition 2.

$\left(\Rightarrow \mathrm{ord}_{s=1} L(E/K, s) = 1 \right).$

By Kolyvagin & Gross-Zagier,

$$E(K) \otimes \mathbb{Q} = E(\mathbb{Q}) \otimes \mathbb{Q} = \mathbb{Q}\, \underset{\substack{\uparrow \\ \text{Heegner pt.}}}{y_K} \cong \mathbb{Q}$$

& $\# Ш(E/K) < \infty$.

By Kobayashi's p-adic GZ formula,

$$\mathcal{L}'_{p,\alpha}(E/K)(0) = \frac{1}{c_E^2 u_K^2} \cdot \left(1 - \frac{1}{\alpha}\right)^4 \cdot \underset{\substack{\uparrow \\ \text{p-adic ht.}}}{\langle y_K, y_K \rangle_{p,\alpha}}$$

where $\mathcal{L}_{p,\alpha}(E/K) := \mathcal{L}_{p,\alpha}(E) \cdot \mathcal{L}_{p,\alpha}(E^K) \cdot \frac{\Omega_E \cdot \Omega_{E^K}}{\Omega_{E/K}}$

$\in \mathbb{Q}_p[\![\Gamma]\!]$.

$$\underset{\substack{\uparrow \\ \text{GZ formula} \\ + \\ \text{interp. of } \mathcal{L}_{p,\alpha}(E^K)}}{\Longrightarrow} \mathcal{L}'_{p,\alpha}(E)(0) = \left(1 - \frac{1}{\alpha}\right)^2 \cdot \boxed{\frac{L'(E,1)}{\langle P, P \rangle_\infty \cdot \Omega_E}} \cdot \langle P, P \rangle_{p,\alpha}$$

where P = gen. of $E(\mathbb{Q})/\text{tors}$.

While by Kobayashi's MC + Perrin's work,

$$\mathcal{L}'_{p,\alpha}(E)(0) \sim_p \left(1 - \frac{1}{\alpha}\right)^2 \cdot \boxed{\#Ш(E/\mathbb{Q}) \cdot \prod_{\ell | N} c_\ell(E/\mathbb{Q})} \cdot \langle P, P \rangle_{p,\alpha}$$

$$\underset{\langle P,P \rangle_{p,\alpha} \neq 0}{\Longrightarrow} \text{p-part of BSD formula holds.} \quad \square$$