$\boxed{\text{Lecture C2}}$ "Cuspidal Hida theory for semi-ordinary forms"

§ 1. Setting

K = imaginary quadratic field where $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits

$G = GU(3,1)$ $\qquad G(\mathbb{Q}_p) \simeq GL_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$

• Integral moduli problems: For fixed $K_f^p \subset G(\mathbb{A}_f^p)$ neat compact open

moduli of $\{\underline{A} = (A, \lambda, \iota, \alpha^p)\}$

$$\begin{tikzcd} & & \mathcal{G} \arrow[d] & \\ S \arrow[rr, hook] & & S^{tor} \arrow[u, bend left, "e"] \arrow[r, "\pi"] & S^{min} \end{tikzcd} \quad /\mathcal{O}_{K,\mathfrak{p}}$$

$\mathcal{G}$ ← univ. semi-abelian scheme

$\underline{\omega} := e^*\Omega^1_{\mathcal{G}/S^{tor}}$, $\qquad \omega = \det \underline{\omega}$

$E \in H^0(S^{tor}, \omega^{t_E(p-1)})$ $(t_E > 0)$ lift of Hasse invariant.

• Levels at p:

$$K^1_{p,n} := \left\{ \gamma \equiv \begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \bmod p^n \right\}$$

$\cap$

$$K^0_{p,n} := \left\{ \gamma \equiv \begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \bmod p^n \right\} \subset G(\mathbb{Z}_p).$$

- Igusa tower:

moduli of $\{(\underline{A}, \alpha_p)\}$

$$\begin{tikzcd}
\text{level } K^1_{p,n}: & \mathcal{T}_n \arrow[r, hook] & \mathcal{T}_n^{\mathrm{tor}} \arrow[r] & S^{\mathrm{tor}}[\tfrac{1}{E}] \\
\text{red'n mod } p^m & \mathcal{T}_{n,m} \arrow[r, hook] \arrow[u] & \mathcal{T}_{n,m}^{\mathrm{tor}} \arrow[r] \arrow[u] & S_m^{\mathrm{tor}}[\tfrac{1}{E}] \arrow[u]
\end{tikzcd}$$

$$\circlearrowleft$$

$$T_{SO}(\mathbb{Z}_p) = \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \twoheadrightarrow K^1_{p,n}/K^0_{p,n}$$

level $K^0_{p,n}$: $\mathcal{T}_n^0$, $\mathcal{T}_n^{0,\mathrm{tor}}$, etc.

- Mod p^m automorphic forms on G:

$$V_{n,m} := H^0(\mathcal{T}_{n,m}^{\mathrm{tor}}, \mathcal{O})$$

$$\cup$$

cuspidal: $$V^0_{n,m} := H^0(\mathcal{T}_{n,m}^{\mathrm{tor}}, \mathcal{I})$$

$$(\mathcal{T}_{n,m}^{\mathrm{tor}} \to S_m^{\mathrm{tor}}[\tfrac{1}{E}])^* \underbrace{\mathcal{I}_{S^{\mathrm{tor}}}}_{\text{ideal sheaf of } \partial S^{\mathrm{tor}}}$$

- Classical embeddings:

$$H^0(S^{tor}, \omega_{\underline{t}}) \hookrightarrow \varprojlim_m \varinjlim_n V_{n,m}[t^+,t^-].$$

$\uparrow$ weight $\underline{t} = (0,0,t^+;t^-)$

$$H^0(S^{tor}, \omega_{\underline{t}} \otimes \mathcal{I}) \hookrightarrow \varprojlim_m \varinjlim_n V^0_{n,m}[t^+,t^-]$$

$$M_{\underline{t}}(K_f^p K^1_{p,n}) \hookrightarrow \left(\varprojlim_m \varinjlim_n V_{n,m}[t^+,t^-]\right)\left[\tfrac{1}{p}\right]$$

$$M^0_{\underline{t}}(K_f^p K^1_{p,m}) \hookrightarrow \left(\varprojlim_m \varinjlim_n V^0_{n,m}[t^+,t^-]\right)\left[\tfrac{1}{p}\right].$$

Theorem (Cuspidal Hida theory for semi-ord forms).

Put

- $\mathcal{V}^0 := \varinjlim_m \varinjlim_n V^0_{n,m} \circlearrowleft \tilde{\Lambda}_{so} = \mathbb{Z}_p[\![T_{so}(\mathbb{Z}_p)]\!]$

$$\tilde{\Lambda}_{so} \twoheadrightarrow \Lambda_{so}$$

- $\mathcal{V}^{0,*}_{so} := \mathrm{Hom}\left(e_{so}\mathcal{V}^0, \mathbb{Q}_p/\mathbb{Z}_p\right),$

  where $e_{so} = \lim_{r\to\infty} U_p^{r!}$

  $\left(U_p \leftrightarrow \begin{pmatrix} p^2 & & & \\ & p^2 & & \\ & & p & \\ & & & p^{-1} \end{pmatrix}\right)$

- $\mathcal{M}^0_{so} := \mathrm{Hom}_{\Lambda_{so}}\left(\mathcal{V}^{0,*}_{so}, \Lambda_{so}\right)$

$$\mathcal{M}^0_{so} = \left\{ \begin{array}{c} \Lambda_{so}\text{-adic} \\ \text{cuspidal} \\ \text{semi-ord forms} \end{array} \right\}$$

Then:

(1) $\mathcal{V}_{so}^{0,*}$ = free of finite rank $/\Lambda_{so}$.

(2) $\forall\, (t^+, t^-) \in \mathrm{Hom}_{cts}(T_{so}(\mathbb{Z}_p), \mathbb{Q}_p^\times)$,

we have isomorphisms

$$\mathcal{M}^0_{so} \underset{\tilde{\Lambda}_{so}}{\otimes} \tilde{\Lambda}_{so}/\mathrm{Ker}(t^+, t^-) \xrightarrow{\sim} \Big(\varprojlim_m \varinjlim_n e_{so} V^0_{n,m} \Big)[t^+, t^-].$$

with $\underline{t} = (0, 0, t^+; t^-)$ dominant

(3) $\forall\, (t^+, t^-)$ as in (2), there are embeddings

$$e_{so} M^0_{\underline{t}}(K_f^p K^1_{p,n}) \hookrightarrow \Big(\mathcal{M}^0_{so} \underset{\tilde{\Lambda}_{so}}{\otimes} \tilde{\Lambda}_{so}/\mathrm{Ker}(t^+, t^-) \Big) \otimes \mathbb{Q}_p,$$

and given $0 \leqslant -t^+$,

this is an isomorphism for $t^- \gg -t^+$.

<u>Remark</u>. The method of proof parallels standard Hida theory.

§2. A key ingredient: Base-change property

Proposition 1. Reduction mod p^m defines

$$H^0(\mathcal{J}_n^{0,tor}[\tfrac{1}{E}], \omega_{\underline{t}} \otimes \mathcal{I}) \otimes \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\sim} H^0(\mathcal{J}_{n,m}^{0,tor}, \omega_{\underline{t}} \otimes \mathcal{I}).$$

Proof. We have $R^1\pi_{n,*}(\omega_{\underline{t}} \otimes \mathcal{I}) = 0$,

$$\pi_n : \mathcal{J}_n^{0,tor} \to \mathcal{J}_n^{0,min}$$

$$\Rightarrow 0 \to \pi_{n,*}(\omega_{\underline{t}} \otimes \mathcal{I}) \xrightarrow{p^m} \pi_{n,*}(\omega_{\underline{t}} \otimes \mathcal{I}) \to \pi_{n,*}(\omega_{\underline{t}} \otimes \mathcal{I}_m) \to 0$$

exact seq. of sheaves on $\mathcal{J}_n^{0,tor}$

Taking global section & using $\mathcal{J}_n^{0,min}[\frac{1}{E}]$ affine gives the result $\boxtimes$

§ 3. Semi-ordinary projector

Proposition 2 The limit

$$e_{so} = \lim_{r\to\infty} U_p^{r!}$$

converges in $H^0(\mathcal{T}_{n,m}^{0,tor}, \omega_{\underline{t}} \otimes I)$ and in $\mathcal{V}^0$.

Proof. Let $\vec{f}_m \in H^0(\mathcal{T}_{n,m}^{0,tor}, \omega_{\underline{t}} \otimes I)$,

and take $\vec{f} \in H^0(\mathcal{T}_n^{0,tor}[\frac{1}{E}], \omega_{\underline{t}} \otimes I)$ lift of $\vec{f}_m$

$\Rightarrow$ for $\ell \gg 0$, $\vec{f} E^\ell \in H^0(\mathcal{T}_n^{0,tor}, \omega_{\underline{t}+\ell t_E(p-1)} \otimes I)$

$$\cap$$

$$M^0_{\underline{t}+\ell t_E(p-1)}(K_f^p K^0_{p,n})$$

$\lim_{r\to\infty} U_p^{r!}$ exists here

Since $\vec{f} E^\ell \equiv \vec{f}_m \mod p^m$, first part follows.

For $\mathcal{V}^\circ$, ETS e_{so} exists in every $V^\circ_{n,m}$

and this follows from

$$\bigoplus_{\underline{t}} \underbrace{H^0(S^{tor}, \omega_{\underline{t}} \otimes I) \otimes \mathbb{Q}_p}_{\substack{\uparrow \\ \lim\limits_{r\to\infty} U_p^{r!} \text{ exists here}}} \hookrightarrow \Big(\varprojlim_m \varinjlim_n \underbrace{H^0(\tilde{\mathcal{J}}_{n,m}, \omega_{\underline{t}} \otimes I)}_{\substack{\cup \\ V^\circ_{n,m}}} \Big)\left[\tfrac{1}{p}\right]$$

with dense image. $\boxed{/}$

Important note. By Coleman theory, given $B \geq 0$

$\dim e_{so} M^\circ_{\underline{t}}(K_f^f K^\circ_{p,n})$ for $\underline{t} = (t_1^+, t_2^+, t_3^+; t_1^-)$

with $t_1^+ - t_2^+ \leq B$

is uniformly bounded,

and the above argument (in first part) shows $\dim_{\mathbb{F}_p} e_{so} H^0(\mathcal{J}^{0,tor}_{1,1}, \omega_{\underline{t}} \otimes I)$

$\wedge$ $\forall \underline{t}$

∞ . $(0,0,t_3^+; t^-)$.

applied to $\vec{f}_1^{(1)}, \dots, \vec{f}_1^{(d)}$ l.i. in

Corollary 1 For any $(t^+, t^-) \in T_{so}(\mathbb{Z}_p)$,

$$\mathcal{V}_{so}^{0,*} \otimes_{\Lambda_{so}} \Lambda_{so}/\ker(t^+, t^-)$$

is p-torsion free.

Proof. We have

$$\begin{array}{ccl} \left(\mathcal{V}_{so}^{0,*} \otimes_{\Lambda_{so}} \Lambda_{so}/\ker(t^+,t^-)\right)^{*} & & \leftarrow \text{Pontryagin dual} \\ \| & & \\ \varinjlim_m \varinjlim_n e_{so} V_{n,m}^{0}[t^+, t^-] & & \\ \uparrow \cong & & \left(\begin{array}{l}\text{by proj.} \\ \text{to values at} \\ \text{highest wt vector}\end{array}\right) \\ \varinjlim_m \varinjlim_n e_{so} H^0(\mathcal{J}_{n,m}^{0,tor}, \omega_{\underline{t}} \otimes \mathcal{I}) & & \\ \uparrow \cong & & (\text{by Prop. 1}) \\ \varinjlim_n e_{so} H^0(\mathcal{J}_n^{0,tor}, \omega_{\underline{t}} \otimes \mathcal{I}) \otimes \mathbb{Q}_p/\mathbb{Z}_p & & \end{array}$$

$\therefore$ divisible. ▨

<u>Corollary 2</u> For any max'l ideal $\mathfrak{m} \subset \tilde{\Lambda}_{so}$,

$$\dim_{\mathbb{F}_p}\left(\mathcal{V}^{0,*}_{so,\mathfrak{m}} \underset{\Lambda_{so}}{\otimes} \Lambda_{so}/(p, T^+, T^-) \right) < \infty$$

$(\Lambda_{so} \cong \mathbb{Z}_p[[T^+, T^-]])$

<u>Proof</u>. We have

$$\left(\mathcal{V}^{0,*}_{so,\mathfrak{m}} \underset{\Lambda_{so}}{\otimes} \Lambda_{so}/(p, T^+, T^-) \right)^*$$

$$\|$$

$$\varinjlim_m \varinjlim_n e_{so} H^0(\mathfrak{T}^{0,tor}_{n,m}, \omega_{\underline{t}} \otimes \mathcal{I})[p]$$

(by "contraction property" of U_p) $\quad \|$ $\quad$ (for any $\ker(\underline{t}^+, \underline{t}^-) \subset \mathfrak{m}$)

$$\varinjlim_m e_{so} H^0(\mathfrak{T}^{0,tor}_{1,m}, \omega_{\underline{t}} \otimes \mathcal{I})[p]$$

(by Prop. 1) $\quad \|$

$$e_{so} H^0(\mathfrak{T}^{0,tor}_{1,1}, \omega_{\underline{t}} \otimes \mathcal{I})$$

$\therefore$ finite-dim'l $/\mathbb{F}_p$ $\square$

§4. Proof of Theorem

Part (1). Fix $\mathfrak{m} \subset \tilde{\Lambda}_{so}$ any maximal ideal.

ETS $\mathcal{V}^{0,*}_{so,\mathfrak{m}}$ = free of finite rank $/\Lambda_{so}$.

By Cor. 2,

$$\underbrace{\dim_{\mathbb{F}_p}}_{d\,=:}\left(\mathcal{V}^{0,*}_{so,\mathfrak{m}} \otimes_{\Lambda_{so}} \Lambda_{so}/(p, T^+, T^-)\right) < \infty \quad (\star)$$

$$\Rightarrow \mathcal{V}^{0,*}_{so,\mathfrak{m}} = \mathrm{Span}_{\Lambda_{so}}(F_1, \dots, F_d).$$

Supp. $a_1 F_1 + \cdots + a_d F_d = 0 \quad (a_i \in \Lambda_{so})$.

By Cor. 1,

$$\mathcal{V}^{0,*}_{so,\mathfrak{m}} \otimes_{\Lambda_{so}} \Lambda_{so}/\mathrm{Ker}(t^+, t^-)$$

is p-torsion free $\forall (t^+, t^-) \in T_{so}(\mathbb{Z}_p)$ w/ $\mathrm{Ker}(t^+, t^-) \overset{\mathfrak{m}}{\cup}$

and from $(\star)$, $\left(\mathcal{V}^{0,*}_{so,\mathfrak{m}} \otimes_{\Lambda_{so}} \Lambda_{so}/\mathrm{Ker}(t^+, t^-)\right) \otimes \mathbb{Z}/p\mathbb{Z}$

$\cong \mathbb{F}_p^d$

$$\Rightarrow \mathcal{V}^{0,*}_{so,m} \underset{\Lambda_{so}}{\otimes} \Lambda_{so}/\ker(t^+,t^-) \cong \mathbb{Z}_p^d$$

$$\Rightarrow a_1, \dots, a_d \in \bigcap_{(t^+,t^-)} \ker(t^+,t^-) = 0$$

$$\therefore \mathcal{V}^{0,*}_{so,m} = \text{finite free } / \Lambda_{so}.$$

Part (2): From part (1),

$$\mathcal{M}^0_{so} \underset{\Lambda_{so}}{\otimes} \tilde{\Lambda}_{so}/\ker(t^+,t^-)$$

$$\cong \mathrm{Hom}\Big[\Big(\underbrace{\varinjlim_m \varinjlim_n e_{so} V^0_{n,m}[t^+,t^-]}_{p\text{-divisible}}\Big)^*, \mathbb{Z}_p\Big]$$

$$\cong \mathrm{Hom}\Big[\mathrm{Hom}\big(\varprojlim_m \varinjlim_n e_{so} V^0_{n,m}[t^+,t^-], \mathbb{Z}_p\big), \mathbb{Z}_p\Big]$$

$$\cong \varprojlim_m \varinjlim_n e_{so} V^0_{n,m}[t^+,t^-].$$

Part (3): By part (2),

ETS given $0 \leq -t^+$, the classical embedding

$$e_{so} M^0_{\underline{t}}(K^p_f K^1_{p,n}) \hookrightarrow \underbrace{\overbrace{\left(\varprojlim_m \varinjlim_n e_{so} V^0_{n,m}[t^+,t^-]\right)}^{V^0_{so}[t^+,t^-]}}_{} \otimes \mathbb{Q}_p$$

is isomorphism for $t^- \gg -t^+$.

$$\| \leftarrow \text{as in Cor. 1}$$

$$\varprojlim_m e_{so} H^0(\mathcal{J}^{0,tor}_{1,m}, \omega_{\underline{t}} \otimes \mathcal{I})$$

p-torsion free

$$\Rightarrow \dim_{\mathbb{F}_p}\left(V^0_{so}[t^+,t^-] \otimes \mathbb{Z}/p\mathbb{Z}\right)$$

$$\| \leftarrow \text{as in Cor 2.}$$

$$\dim_{\mathbb{F}_p} e_{so} H^0(\mathcal{J}^{0,tor}_{1,1}, \omega_{\underline{t}} \otimes \mathcal{I})$$

$$\| \leftarrow \text{by base-change property + p-torsion free}$$

$$\dim_{\mathbb{Q}_p} e_{so} H^0(\mathcal{J}^{0,tor}_1[\tfrac{1}{E}], \omega_{\underline{t}} \otimes \mathcal{I}) \otimes \mathbb{Q}_p$$

$$\overset{!!}{\cdot\cdot}$$

$$d$$

$$\Longrightarrow V^{0}_{so}[t^{+}, t^{-}] \simeq \mathbb{Z}_p^{d}$$

Finally, show $\dim_{\mathbb{Q}_p} e_{so} M^{0}_{\underline{t}}(K_f^{p} K^{1}_{p,n}) \geqslant d$ for $t^{-} \gg -t^{+}$

by multiplying a basis of $V^{0}_{so}[t^{+}, t^{-}]$ by $E \cdot (\chi \circ \det)$

suitable
unr. Hecke char.
of K.

$\boxed{/\!/}$