Lecture C3 "Proof of signed Main Conjectures"

First day: For $E/\mathbb{Q}$ elliptic curve, $\mathrm{cond}\, E = N$

$p \nmid 2N$ supersingular prime with $a_p = 0$

$K/\mathbb{Q}$ suitable imag. quadr. field,

we showed.

- Kobayashi's Main Conj. $\Rightarrow$ p-part of BSD formula in rank ≤ 1
- Signed Heegner Point Main Conj. $\Rightarrow$ p-converse to Gross-Zagier & Kolyvagin.

Today: Explain how

("Main Theorem" in Xin's first talk) $\Rightarrow$ Kobayashi's Main Conj. & Signed Heegner Point Main Conj.

Recall:

"Main Theorem"

Let $f \in S_2(\Gamma_0(N))$, $p \nmid 2N$ prime,

$K/\mathbb{Q}$ imag. quadr. field where $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits.

Suppose:

(i) N is $\square$-free

(ii) $\exists\, q \| N$ non-split in K

(iii) If N is odd, then 2 splits in K

(iv) $\bar{\rho}_f|_{G_K}$ abs. irred.

Then

$$\operatorname{length}_{\mathfrak{P}} X_{\mathfrak{p}}(A_f/K_\infty) \geq \operatorname{ord}_{\mathfrak{P}} \mathcal{L}_{\mathfrak{p}}(A_f/K)$$

$\forall\, \mathfrak{P} \subset \Lambda_K$ ht 1 prime

with $\mathfrak{P} \neq \mathfrak{P}_0 \Lambda_K$ for some $\mathfrak{P}_0 \subset \Lambda$

Here:

- $A_f = GL_2$-type ab. var$/\mathbb{Q}$ assoc. to f

  (we'll take f s.t. $A_f \sim E$).

- $X_{\mathfrak{p}}$ has $\begin{cases} \text{relaxed condition at } w \mid \mathfrak{p} \\ \text{strict condition at } w \mid \bar{\mathfrak{p}} \end{cases}$

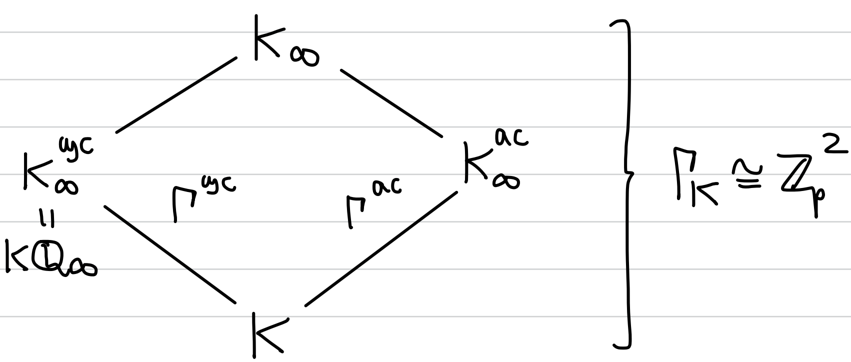
  $X_{\mathfrak{p}} = X^{\text{rel},\text{str}}$

- $\mathcal{L}_{\mathfrak{p}}(A_f/K) = \mathrm{Tw}_{\xi^{-1}}\left(\mathcal{L}^{\Sigma}_{f,\chi,\xi}\right)$ made primitive.

  $\in \Lambda_K.$

§ 1. Two variable signed Main Conj.

$K/\mathbb{Q}$ imag. quadr. where $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits.



<u>B.-D. Kim</u>: $\exists$ four doubly-signed Selmer gps.

$$\mathrm{Sel}_{p^\infty}^{\pm,\pm}(E/K_\infty) := \ker\left[\mathrm{Sel}_{p^\infty}(E/K_\infty) \longrightarrow \frac{E(K_{\infty,\mathfrak{p}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}{E^{\pm}(K_{\infty,\mathfrak{p}})} \times \frac{E(K_{\infty,\bar{\mathfrak{p}}}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}{E^{\pm}(K_{\infty,\bar{\mathfrak{p}}})} \right]$$

<u>Loeffler</u>: $\exists$ four doubly-signed p-adic L-f'ns

$$\mathcal{L}_p^{+,+}(E/K), \ldots, \mathcal{L}_p^{-,-}(E/K) \in \Lambda_K$$

decomposing four unbounded distributions

$$\underbrace{\mathcal{L}_p^{\alpha,\alpha}(E/K), \ldots, \mathcal{L}_p^{\bar{\alpha},\bar{\alpha}}(E/K)}_{} \in \mathbb{Q}_p[\![\Gamma_K]\!] \qquad (\alpha, \bar{\alpha} = \pm\sqrt{-p}).$$

↑ constructed using generalised Mazur-Tate elts for $GL_{2/K}$.

<u>Conjecture</u> (B.-D. Kim).

$$X^{\pm,\pm}(E/K_\infty) := \mathrm{Sel}_{p^\infty}^{\pm,\pm}(E/K_\infty)^\wedge \text{ is } \Lambda_K\text{-torsion},$$

with

$$\mathrm{char}_{\Lambda_K} X^{\pm,\pm}(E/K_\infty) = \left(\zeta_p^{\pm,\pm}(E/K)\right).$$

§2. Beilinson–Flach classes

<u>Theorem</u> (after $\left.\begin{matrix}\text{Lei}\\ \text{Kings}\end{matrix}\right\}$ – Loeffler – Zerbes).

For each $\lambda, \mu \in \{\alpha, \bar{\alpha}\}$,

$\exists$ 2-variable Beilinson–Flach class

$$BF^{\lambda} \in H^1_{Iw}(K_\infty, T_pE) \otimes_{\Lambda_K} \mathbb{Q}_p[[\Gamma_K]]$$

together with 2 "Explicit Reciprocity Laws":

(ERL1) $\mathrm{res}_{\bar{\mathfrak{p}}}(BF^{\lambda}) \overset{Col^{\mu}}{\longmapsto} \mathcal{L}_p^{\lambda,\mu}(E/K)$ (Loeffler's)

$$\Vert$$

$$\langle \mathcal{L}^{PR}(-), \omega_g \otimes \eta_{f^{\mu}} \rangle$$

(ERL2) $\mathrm{res}_{\mathfrak{p}}(BF^{\lambda}) \overset{\widetilde{Log}^{\mu}}{\longmapsto} \mathcal{L}_{\mathfrak{p}}(E/K)$ (as in "Main Theorem")

$$\Vert$$

$$h_K \cdot \mathcal{L}_{\mathfrak{p}}^{ac} \cdot \underbrace{\langle \mathcal{L}^{PR}(-), \eta_g \otimes \omega_{f^{\mu}} \rangle}_{=: Log^{\mu}}$$

$\mathcal{L}_{\mathfrak{p}}^{ac}$ ← anti-cycl. Katz

<u>Proof</u>. Follows from BF classes over $\mathbb{Q}_\infty/\mathbb{Q}$ assoc. to $f \otimes g$

g ← CM Hida family w/ $g_1 = \mathrm{Eis}(1, \eta_{K/\mathbb{Q}})$

+ a result of Burungale-Skinner-Tian

$V_g \cong \mathrm{Ind}_K^{\mathbb{Q}} \Psi_{\mathfrak{p}}$ ← univ. character for $K_\infty^{\mathfrak{p}}/K$

$K_\infty^{\mathfrak{p}}$ ← max'l subext. of K_∞ where $\mathfrak{p}$ is unr.

+ the ERLs in Kings-Loeffler-Zerbes ∎

<u>Wan</u>: $\exists\ \Lambda_K$-module isomorphism

(1) $$\mathrm{Col}^{\pm} : \left. \frac{H^1_{Iw}(K_{\infty,\bar{\mathfrak{p}}}, T_pE)}{H^1_{\pm}(K_{\infty,\bar{\mathfrak{p}}}, T_pE)} \right\} = H^1_{/\pm} \xrightarrow{\ \cong\ } \Lambda_K$$

$H^1_{\pm}(K_{\infty,\bar{\mathfrak{p}}}, T_pE)$: exact annihilator of $E^{\pm}(K_{\infty,\bar{\mathfrak{p}}})$ under local Tate duality

(2) $$\mathrm{Log}^{\pm} : H^1_{\pm}(K_{\infty,\mathfrak{p}}, T_pE) \xrightarrow{\ \cong\ } \Lambda_K$$

s.t. $$\begin{pmatrix} \mathrm{Col}^{\alpha} \\ \mathrm{Col}^{\bar{\alpha}} \end{pmatrix} = M_{\log,\bar{\mathfrak{p}}} \begin{pmatrix} \mathrm{Col}^{+} \\ \mathrm{Col}^{-} \end{pmatrix}$$

& $$\begin{pmatrix} \mathrm{Log}^{\alpha} \\ \mathrm{Log}^{\bar{\alpha}} \end{pmatrix} = M_{\log,\mathfrak{p}} \begin{pmatrix} \mathrm{Log}^{+} \\ \mathrm{Log}^{-} \end{pmatrix}$$

(3) bounded BF classes $BF^{\pm} \in H^1_{Iw}(K_\infty, T_pE)$

s.t.

$$\begin{pmatrix} BF^{\alpha} \\ BF^{\bar{\alpha}} \end{pmatrix} = M_{\log,\wp} \begin{pmatrix} BF^{+} \\ BF^{-} \end{pmatrix}.$$

Corollary 1.

(ERL1)' $\quad \mathrm{Col}^{+}(\mathrm{res}_{\bar{\wp}}(BF^{+})) = \mathcal{L}_p^{+,+}(E/K).$

(ERL2)' $\quad \widetilde{\mathrm{Log}}^{+}(\mathrm{res}_{\wp}(BF^{+})) = \mathcal{L}_{\wp}(E/K).$

$$\widetilde{\mathrm{Log}}^{+} = h_K \cdot \mathcal{L}_p^{ac} \cdot \mathrm{Log}^{+}$$

Corollary 2 For any ht 1 prime $\mathfrak{P} \subset \Lambda_K$, TFAE:

(1) $\ell t_{\mathfrak{P}}\, X^{+,+}(E/K_\infty) \geqslant \mathrm{ord}_{\mathfrak{P}}\, \mathcal{L}_p^{+,+}(E/K).$

(2) $\ell t_{\mathfrak{P}}\, X^{+,\mathrm{str}}(E/K_\infty) \geqslant \ell t_{\mathfrak{P}} \left(\dfrac{\check{S}_p^{+,\mathrm{rel}}(K_\infty, T_pE)}{(BF^{+})} \right).$

(3) $\ell t_{\mathfrak{P}}\, X_{\wp}(E/K_\infty) \geqslant \mathrm{ord}_{\mathfrak{P}}\, \mathcal{L}_{\wp}(E/K).$

Proof. Immediate from Cor. 1 + double application of Poitou-Tate duality:

$$
\begin{array}{ccccc}
 & & \Lambda_K/(\mathcal{L}_p^{+,+}(E/K)) & & \\
 & & \cong\uparrow \ \text{via (ERL1)}' & & \\
 & & \dfrac{H^1_{/+}(K_{\infty,\bar{\mathfrak{p}}},T_pE)}{(\mathrm{res}_{\bar{\mathfrak{p}}}(BF^+))} & \longrightarrow & X^{+,+}(E/K_\infty) \\
 & \overset{\mathrm{res}_{\bar{\mathfrak{p}}}}{\nearrow} & & & \searrow \\
0 \to \dfrac{\check{S}_p^{+,\mathrm{rel}}(K_\infty,T_pE)}{(BF^+)} & & & & X^{+,\mathrm{str}}(E/K_\infty) \to 0 \\
 & \underset{\mathrm{res}_{\mathfrak{p}}}{\searrow} & & & \nearrow \\
 & & \dfrac{H^1_{+}(K_{\infty,\mathfrak{p}},T_pE)}{(\mathrm{res}_{\mathfrak{p}}(BF^+))} & \longrightarrow & \underbrace{X^{\mathrm{rel},\mathrm{str}}(E/K_\infty)} \\
 & & \cong\downarrow \ \text{via (ERL2)}' & & \| \\
 & & \Lambda_K/(\mathcal{L}_{\mathfrak{p}}(E/K)) & & X_{\mathfrak{p}}(E/K_\infty)
\end{array}
$$

∎

§3. Proof of Kobayashi's Main Conj.

As before, $E/\mathbb{Q}$ elliptic curve, cond $= N$

$p \nmid 2N$ supersingular with $a_p = 0$.

<u>Theorem</u> (Wan).

Suppose E is semistable (i.e., N is $\square$-free).

Then Kobayashi's Main Conj. holds.

<u>Proof</u>. By Ribet's level lowering results,

$\exists$ odd prime $q \| N$ s.t. $E[p]$ ramif. at q.

Choose $K/\mathbb{Q}$ imag. quadr. s.t.

- $p = \mathfrak{p}\bar{\mathfrak{p}}$ } splits in K
- every prime $\ell \mid 2N/q$ } splits in K
- q inert in K.

By "Main Thm" + Cor. 2

$$\ell t_{\mathfrak{P}} X^{+,+}(E/K_\infty) \geq \mathrm{ord}_{\mathfrak{P}} \mathcal{L}_p^{+,+}(E/K)$$

$\forall \mathfrak{P} \subset \Lambda_K$ ht 1 prime with $\mathfrak{P} \neq \mathfrak{P}_0 \Lambda_K$ for some $\mathfrak{P}_0 \subset \Lambda$.

But $\mu\left(\mathcal{L}_p^{+,+}(E/K)\big|_{\Gamma^{ac}}\right) = 0$ by Pollack-Weston & Vatsal

so above $\geq$ holds $\forall \mathfrak{P} \subset \Lambda_K$ ht 1 prime.

$$\Longrightarrow \mathrm{char}_\Lambda\left(X^{+,+}(E/K_\infty)/(\gamma^{ac}-1)\right) = \mathrm{char}_\Lambda(X^+(E/\mathbb{Q}_\infty)) \times \mathrm{char}_\Lambda(X^+(E^K/\mathbb{Q}_\infty)).$$

(descend to K_∞^{ac})

$$\cap|$$

$$\left(\mathcal{L}_p^{+,+}(E/K) \bmod \gamma^{ac}-1\right)$$

$$\|$$

$$\left(\mathcal{L}_p^+(E) \cdot \mathcal{L}_p^+(E^K)\right)$$

$\Longrightarrow$ Kobayashi's Main Conj. holds. ☒

(Kobayashi's opposite div. for E & E^K)

§4. Proof of Signed Heegner Point Main Conj.

Let $E/\mathbb{Q}$ & $p \nmid 2N$ as in §3,

and $K/\mathbb{Q}$ imag quadr. field

satisfying Heegner hypothesis

& s.t. $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K.

Theorem (C.–Wan).

Suppose:

(i) E is semistable

(ii) $\exists\, q \| N$ non-split in K.

(iii) If N is odd, then 2 splits in K.

Then the signed Heegner Point Main Conj. holds.

<u>Proof</u>. By the "Main Theorem"

$$\ell t_{\mathfrak{P}} X_{\wp}(E/K_\infty) \geq \mathrm{ord}_{\mathfrak{P}} \mathcal{L}_{\wp}(E/K)$$

$\forall \mathfrak{P} \subset \Lambda_K$ ht 1 prime

with $\mathfrak{P} \neq \mathfrak{P}_0 \Lambda_K$ for some $\mathfrak{P}_0 \subset \Lambda$.

But comparing interpolations:

$$\mathcal{L}_{\wp}(E/K)\big|_{\Gamma^{ac}} = \mathcal{L}_{\wp}^{BDP}(E/K)^2 \quad \text{up to units.}$$

↑ has $\mu = 0$ (Hsieh, Burungale)

$\Rightarrow$ above $\geq$ holds $\forall \mathfrak{P} \subset \Lambda_K$ ht 1 prime.

$$\Rightarrow \mathrm{Char}_{\Lambda^{ac}}\left(X_{\wp}(E/K_\infty) / (\gamma^{cyc} - 1) \right)$$

$$\cap\!\!|$$

$$\left(\mathcal{L}_{\wp}^{BDP}(E/K)^2 \right)$$

(descend to K_∞^{ac})

By an extension of BDP's formula:

$$\mathrm{Log}^{+}\left(\mathrm{res}_p(\kappa_\infty^{+})\right) = \mathcal{L}_p^{BDP}(E/K)$$

$$\Longrightarrow \quad \mathrm{char}_{\Lambda^{ac}}\left[X^{+}(E/K_\infty^{ac})_{tors} \right]$$

double applic. of Poitou-Tate duality

$$\cap\!\!\!\mid$$

$$\mathrm{char}_{\Lambda^{ac}}\left(\frac{\check{S}_p^{+}(K_\infty^{ac}, T_pE)}{(\kappa_\infty^{+})} \right)^2$$

$\Longrightarrow$ Signed Heegner Point Main Conj holds. ☒

Kolyvagin system argument to get opposite div.