Non-cuspidal Hida theory for semi-ordinary forms on GU(3,1)

$\mathcal{K}$ imag quad

$p = \mathfrak{p}\bar{\mathfrak{p}}$ splits. $\quad \mathcal{K}_p = \mathcal{K} \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathbb{Q}_p \oplus \mathbb{Q}_p$

$$GU(3,1)(\mathbb{Q}_p) \cong \left\{ (g_1, g_2) \in GL_4(\mathbb{Q}_p)^2 \;\middle|\; g_1 \begin{pmatrix} & & 1 \\ & 3 & \\ -1 & & \end{pmatrix} {}^t g_2 = \nu \begin{pmatrix} & & 1 \\ & 3 & \\ -1 & & \end{pmatrix} \right\}$$
$$\cong GL_4(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$$

$$\begin{matrix} K'_{p,n} & = & \left\{ g \equiv \left(\begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}, * \right) \bmod p^n \right\} \\ \cap & & \\ GU(3,1)(\mathbb{Z}_p) & & \end{matrix}$$

$T^{tor}_{n,m}$ = Igusa tower of level $K'_{p,n}$ over $\mathbb{Z}/p^m\mathbb{Z}$

$$\mathcal{V} = \varinjlim_m \varinjlim_n V_{n,m} = H^0(T^{tor}_{n,m}, \mathcal{O}_{T^{tor}_{n,m}})$$
$$\mathcal{V}^0 = \varinjlim_m \varinjlim_n V^0_{n,m} = H^0(T^{tor}_{n,m}, \mathcal{O}_{T^{tor}_{n,m}}(-D))$$

$\supset \mathbb{Z}_p[[T_{SO}(\mathbb{Z}_p)]]$

$$U_p \leftrightarrow \begin{pmatrix} p^2 & & & \\ & p^2 & & \\ & & p & \\ & & & p^{-1} \end{pmatrix}$$

$\lim_{r\to\infty} U_p^{r!}$ converges on $\mathcal{V}^\circ$, $\quad e_{so} \overset{\text{def}}{=} \lim_{r\to\infty} U_p^{r!}$

$\mathcal{M}_{so}^\circ = \mathrm{Hom}_{\Lambda_{so}}\left(\mathrm{Hom}(e_{so}\mathcal{V}^\circ, \mathbb{Q}_p/\mathbb{Z}_p), \Lambda_{so}\right)$ is free

of finite rk over $\Lambda_{so} = \mathbb{Z}_p[\![T_{so}(1+p\mathbb{Z}_p)]\!] \cong \mathbb{Z}_p[\![T^+, T^-]\!]$

<u>Goal</u>: Define e_{so} on $\mathcal{V}$, $\mathcal{M}_{so}$

prove the so-called fundamental exact seq for studying Klingen Eis congruence :

$$0 \longrightarrow \mathcal{M}_{so}^\circ \longrightarrow \mathcal{M}_{so} \xrightarrow{\Phi} \bigoplus_{\substack{\text{p-adic} \\ \text{cusps of tame level.}}} M_{GU(2)} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\![T_{so}(\mathbb{Z}_p)]\!] \longrightarrow 0$$

(Later, we construct $\mathbb{E}_\varphi^{\text{Kling}} \in \mathcal{M}_{so} \otimes_{\Lambda_{so}} \hat{\mathcal{O}}_L^{ur}[\![\Gamma_K]\!] \otimes \mathbb{Q}$ and show $\Phi(\mathbb{E}_\varphi^{\text{Kling}})$ is divisible by an imprimitive p-adic L-fcn $\mathcal{L}$.

The above exact seq $\Rightarrow \exists\ \mathbb{E}' \in \mathcal{M}_{so} \otimes \mathbb{Q}$, s.t. $\Phi(\mathbb{E}') = \dfrac{\Phi(\mathbb{E}_\varphi^{\text{Kling}})}{\mathcal{L}}$

$\mathbb{E}_\varphi^{\text{Kling}} - \mathcal{L}\,\mathbb{E}' \in \mathcal{M}_{so}^\circ$)

Two possible approaches

(1) Establish Hida theory for $\mathcal{V}$ and then show the exact seq.

(Skinner - Urban, Hsieh)

(2) Obtain $0 \longrightarrow \mathcal{V}^{\circ} \longrightarrow$ a modification of $\mathcal{V}$ $\longrightarrow \bigoplus M_{GU(2)}(\mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathbb{Z}_p[\![T_{SO}(\mathbb{Z}_p)]\!] \longrightarrow 0$

and deduce the wanted results. (Rosso - L., Wan)

A key technical step is to analyze $\mathcal{V}/\mathcal{V}^{\circ}$

$$\pi : T_n^{tor} \longrightarrow T_n^{min}$$

T_n^{min} is affine $\Rightarrow$ $V_{n,m}/V_{n,m}^{\circ}$

$$= H^0\Big(T_{n,m}^{min}, \underbrace{\pi_* \mathcal{O}_{T_{n,m}^{tor}} / \pi_* \mathcal{O}_{T_{n,m}^{tor}}(-D)}_{\bigoplus_{g \in C_n} \mathcal{O}_{Sh_{GU(2),K_g}} \otimes \mathbb{Z}/p^m\mathbb{Z}}\Big)$$

$$= \bigoplus_{g \in C_n} M_{GU(2)}(K_g ; \mathbb{Z}/p^m\mathbb{Z})$$

$T_{SO}(\mathbb{Z}_p)$ acts $M_{GU(2)}(K_g ; \mathbb{Z}/p^m\mathbb{Z})$ and permutes the direct summands.

$$P_D = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ & & & \times \end{pmatrix}$$

$$C_n = \begin{pmatrix} \mathcal{O}_{\mathcal{K},(p)}^{\times} & & \\ & 1 & \\ & & \mathcal{O}_{\mathcal{K},(p)}^{\times} \end{pmatrix} \ltimes P'(\mathbb{A}_f) \Big\backslash GU(3,1)(\mathbb{A}_f^p) \times P_D(\mathbb{Z}_p) \Big/ K_f^p \, K'_{p,n}$$

$P' \subset$ Klingen parabolic

$$\begin{pmatrix} 1 & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & 1 \end{pmatrix} \qquad \begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$$

$$K_{g,p} = \mathrm{Im}\left(g_p K'_{p,n} g_p^{-1} \cap P_D(\mathbb{Z}_p) \longrightarrow GU(2,\mathbb{Z}_p) \right)$$

$$C_n = \begin{pmatrix} \mathcal{O}_{\mathcal{K},(p)}^{\times} & & \\ & 1 & \\ & & \mathcal{O}_{\mathcal{K},(p)}^{\times} \end{pmatrix} \ltimes P'(\mathbb{A}_f^p) \Big\backslash GU(3,1)(\mathbb{A}_f^p) \Big/ K_f^p$$

$$\times \underbrace{\begin{pmatrix} 1 & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & 1 \end{pmatrix} \Big\backslash \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ & & & \times \end{pmatrix} \Big/ \begin{pmatrix} \times & \times & \times & \times \\ p\times & \times & \times & \times \\ p^n\times & p^n\times & 1+p^n\times & \times \\ & & & 1+p^n\times \end{pmatrix}}$$

$$\underbrace{\begin{pmatrix} 1 & \times & \times \\ & \times & \times \\ & \times & \times \end{pmatrix} \Big\backslash GL_3(\mathbb{Z}_p) \Big/ \begin{pmatrix} \times & \times & \times \\ p\times & \times & \times \\ p^n\times & p^n\times & 1+p^n\times \end{pmatrix}} \times (\mathbb{Z}/p^n\mathbb{Z})^{\times}$$

$$\begin{pmatrix} & & (\mathbb{Z}/p^n\mathbb{Z})^\times \\ 1 & & \\ & 1 & \end{pmatrix} \amalg \begin{pmatrix} 1 & & \\ & 1 & \\ \mathbb{Z}/p^n\mathbb{Z} & & 1 \end{pmatrix} \amalg \begin{pmatrix} & 1 & \\ 1 & & \\ & \mathbb{Z}/p^n\mathbb{Z} & 1 \end{pmatrix}$$

$K_{g,p} = \begin{pmatrix} \times & \times \\ p\times & \times \end{pmatrix}$ more complicated

Let $C_n^b = \left\{ g \in C_n \;\middle|\; g_p \in \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ & & & \times \end{pmatrix} \right\} \subset P_D(\mathbb{Z}_p)$, with the middle 2×2 block in $GL_2(\mathbb{Z}_p)$

$= C^p \times T_{SO}(\mathbb{Z}/p^n\mathbb{Z})$

$V_{n,m}^b \subset V_{n,m}$ vanish along strata indexed by $g \notin C_n^b$

$$\mathcal{V}^b = \varinjlim_m \varinjlim_n V_{n,m}^b$$

Then $0 \to \mathcal{V}^0 \to \mathcal{V}^b \xrightarrow{\Phi} \bigoplus_{g \in C^p} M_{GU(2)}\left(\begin{pmatrix} \times & \times \\ p\times & \times \end{pmatrix}; \mathbb{Q}_p/\mathbb{Z}_p\right) \otimes \mathbb{Z}_p⟦T_{SO}(\mathbb{Z}_p)⟧ \to 0$

We show
- $\mathcal{V}^b$ is U_p-stable
- $U_p^m V_{n,m} \subset V_{n,m}^b$
- The above exact seq is essentially U_p-equiv:

  $\exists\, N \geq 1$, s.t. $\Phi(U_p^N F) = \Phi(F)$

$\lim_{r\to\infty} U_p^{r!}$ converges on $\mathcal{V}^{\circ}$ and $M_{GU(2)}\left(\begin{pmatrix} \times & \times \\ p\times & \times \end{pmatrix}; \mathbb{Q}_p/\mathbb{Z}_p\right) \otimes \mathbb{Z}_p[\![T_{so}(\mathbb{Z}_p)]\!]$

$\Rightarrow$ converges on $\mathcal{V}^b$ $\Rightarrow$ converges on $\mathcal{V}$

$$e_{so} \overset{\text{def}}{=} \lim_{r\to\infty} U_p^{r!}$$

Above exact seq $\Rightarrow$

$$0 \to \mathcal{M}_{so}^0 \longrightarrow \mathrm{Hom}_{\Lambda_{so}}\left(\mathrm{Hom}_{\mathbb{Z}_p}(e_{so}\mathcal{V}^b, \mathbb{Q}_p/\mathbb{Z}_p), \Lambda_{so}\right)$$

$$\longrightarrow \bigoplus_{g\in C^p} M_{GU(2)}\left(\begin{pmatrix} \times & \times \\ p\times & \times \end{pmatrix}; \mathbb{Z}_p\right) \otimes \mathbb{Z}_p[\![T_{so}(\mathbb{Z}_p)]\!] \longrightarrow 0$$

$$\mathrm{Hom}_{\Lambda_{so}}\left(\mathrm{Hom}_{\mathbb{Z}_p}(e_{so}\mathcal{V}, \mathbb{Q}_p/\mathbb{Z}_p), \Lambda_{so}\right) \overset{\text{def}}{=} \mathcal{M}_{so}$$

Constructing elts in $\mathcal{M}_{so}$

$T_{so}(1+p\mathbb{Z}_p) \subset \Gamma_{\mathcal{K}}$

Consider $\mathrm{Meas}(T_\chi, V)^{\S}$ $\qquad V = \varprojlim_m \varinjlim_n V_{n,m}$

$\cup$

We can construct μ $\qquad$ $\S$ means equivariant for

by p-adically interpolating $\qquad$ $T_{S_0}(1+p\mathbb{Z}_p)$-action on T_χ, V

q-exp's.

$$e_{S_0}\mu \in \mathrm{Meas}(T; V_{S_0})^{\S} \longrightarrow \mathcal{M}_{S_0}$$

$$V_{S_0} \times \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{V}^b_{S_0}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$$

$$\rightsquigarrow \mathrm{Meas}(T_\chi, V_{S_0}) \times \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{V}^b_{S_0}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathrm{Meas}(T_\chi, \mathbb{Z}_p)$$

$$\cong \mathbb{Z}_p[\![T_\chi]\!]$$

$$\rightsquigarrow \mathrm{Meas}(T_\chi, V_{S_0}) \longrightarrow \mathrm{Hom}_{\Lambda_{S_0}}\left(\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{V}^b_{S_0}, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p[\![T_\chi]\!]\right)$$

$$= \mathcal{M}_{S_0} \otimes_{\Lambda_{S_0}} \mathbb{Z}_p[\![T_\chi]\!].$$

Constructing elts in $\mathcal{M}_{so}$

$\Gamma \supset T_{so}(1+p\mathbb{Z}_p)$ (later $\Gamma = \Gamma_K$)

finite index

Consider $\mathrm{Meas}(\Gamma, V)^{\$}$

We can construct $\underset{\mu}{\in}$ ... $\mu \in$

by p-adically interpolating

q-exp's

$V = \varprojlim_m \varinjlim_n V_{n,m}$

$\$$ means equiv for

$T_{so}(1+p\mathbb{Z}_p)$ - action

on Γ and V

$$e_{so}\mu \in \mathrm{Meas}(\Gamma, V)^{\flat} \longrightarrow \mathcal{M}_{so}$$

$$V_{so} \times \mathrm{Hom}_{\mathbb{Z}_p}(\vartheta^{b}_{so}, \mathbb{Q}_p/\mathbb{Z}_p) \to \mathbb{Z}_p$$

$$\rightsquigarrow \mathrm{Meas}(\Gamma, V_{so}) \times \mathrm{Hom}_{\mathbb{Z}_p}(\vartheta_{so}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathrm{Meas}(\Gamma, \mathbb{Z}_p) \cong \mathbb{Z}_p[\![\Gamma]\!]$$

$$\rightsquigarrow \mathrm{Meas}(\Gamma, V_{so}) \longrightarrow \mathrm{Hom}_{\Lambda_{so}}\left(\mathrm{Hom}_{\mathbb{Z}_p}(\vartheta_{so}, \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Z}_p[\![\Gamma]\!]\right) = \mathcal{M}_{so} \otimes_{\Lambda_{so}} \mathbb{Z}_p[\![\Gamma]\!]$$