The construction of Klingen Eisenstein family and the const term

$\mathcal{K}$ imag quad, p splits

$\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_\infty / \mathcal{K})$

Fix $\xi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \longrightarrow \mathbb{C}^\times$. ∞-type (0, an even integer)

$$\begin{bmatrix} & & 1 \\ & \xi & \\ -1 & & \end{bmatrix} \qquad \xi \qquad \begin{bmatrix} & I_3 \\ -I_3 & \end{bmatrix}$$

$$GU(3,1) \times_{GL(1)} GU(2) \hookrightarrow GU(3,3)$$

$$(g_1, g_2) \longmapsto S^{-1} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} S$$

$$S = \begin{pmatrix} 1 & & & \\ & I_2 & & -\frac{\xi}{2} \\ & & 1 & \\ & -I_2 & & -\frac{\xi}{2} \end{pmatrix}$$

- Construct $\mathbb{E}^{\mathrm{Sieg}} \in \mathrm{Meas}(\Gamma_{\mathcal{K}}, V_{GU(3,3)})$

  $\uparrow$ p-adic forms on $GU(3,3)$

$$\mathbb{E}^{\mathrm{Sieg}} \Big|_{GU(3,1) \times GU(2)} \in \mathrm{Meas}\left(\Gamma_{\mathcal{K}}, V_{GU(3,1)} \otimes M_{GU(2)}\right)$$

$$- \quad \mathbb{E}^{Kling}_{\varphi} = \left\langle e_{so}\, \mathbb{E}^{Sieg}\Big|_{GU(3,1)\times GU(2)} ,\ \varphi \right\rangle_{GU(2)}$$

$$\in Meas\left(\Gamma_{\mathcal{K}},\ V_{GU(3,1),\, so}\right) \qquad (\text{over } \hat{\mathcal{J}}_L^{ur})$$

The construction of $\mathbb{E}^{Sieg}$

Interpolation pts : $\tau \in Hom(\Gamma_{\mathcal{K}}, \overline{\mathbb{Q}}_p^{\times})$ alg s.t.

$\xi\tau$ has ∞-type $(0,k)$, $k \geq 6$ even

For such τ's, choose

$$f_{\xi\tau} \in \underbrace{Ind_Q^{GU(3,3)}(s,\ \xi_0\tau_0)}_{} \Big|_{s=\frac{k-3}{2}} \qquad \xi_0\tau_0 = \xi\tau|\cdot|^{-\frac{k}{2}} \text{ unitary}$$

$I(s, \xi_0\tau_0) = Ind_Q^{GU(3,3)}(s,\ \xi_0\tau_0)$

$$= \bigotimes_v{}' \left\{ f : GU(3,3)(\mathbb{Q}_v) \to \mathbb{C} \;\middle|\; f\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} g\right) = (\xi_0\tau_0)(\det A)\, |\det AD^{-1}|^{\frac{s+3}{2}} f(g) \right\}$$

s.t. $$E^{sieg}_{\xi\tau}(g) = \sum_{\gamma\in Q(\mathbb{Q})\backslash GU(3,3)(\mathbb{Q})} f_{\xi\tau}(\gamma g)$$

has p-adically interpolatable Fourier coeff's as τ among the interpolation pts. and has nonzero semi-ord proj after restriction to $GU(3,1)\times GU(2)$.

The choice at p

$$f_{\xi\tau,p} = M_Q\left(f_{(\xi\tau)^{-c},p}\,(\,\cdot\;\gamma_p)\right) \qquad \gamma_p = \rho_p(S^{-1})$$

$$M_Q : I(-s,(\xi_0\tau_0)^{-c}) \longrightarrow I(s,\xi_0\tau_0)$$

$$(M_Q f)(g) = \int f\left(\begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix} \begin{pmatrix} I_3 & \sigma \\ 0 & I_3 \end{pmatrix} g \right) d\sigma$$

supp on $Q(\mathbb{Q}_p)\begin{pmatrix} & I_3 \\ I_3 & \end{pmatrix} Q(\mathbb{Q}_p)$ big cell

$$f_{(\xi\tau)^{-c},p}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = |\nu|_p^{-s+\frac{3}{2}}\,|\det C^{+}\bar{C}|_p^{s-\frac{3}{2}}\,\xi_0\tau_0(\nu \det C)$$

$$\cdot\,(\mathcal{F}^{-1}\alpha_{\xi\tau})(C^{-1}D)$$

$$C^{-1}D \in \mathrm{Her}_3(\mathcal{K}\otimes\mathbb{Q}_p) \cong M_3(\mathbb{Q}_p)$$

$$\alpha_{\xi\tau}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \mathbb{1}_{M_3(\mathbb{Z}_p)}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

$$\mathbb{1}_{\mathbb{Z}_p^\times}(x_{21}) \cdot \mathbb{1}_{GL_2(\mathbb{Z}_p)}\begin{pmatrix} x_{21} & x_{22} \\ x_{31} & x_{33} \end{pmatrix}$$

$$(\xi\tau)_p^{-1}\left(\det\begin{pmatrix} x_{21} & x_{22} \\ x_{31} & x_{33} \end{pmatrix}\right)$$

This choice gives the correct nebentypus.

Wanted nebentypus at p of restriction to $GU(3,1) \times GU(2)$:

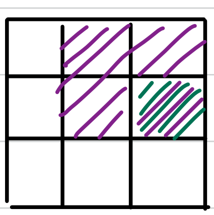
$$\begin{pmatrix} \mathrm{triv} & & & \\ & \mathrm{triv} & & \\ & & (\xi\tau)_{\mathfrak{p}} & \\ & & & (\xi\tau)_{\bar{\mathfrak{p}}}^{-1} \end{pmatrix} \times \begin{pmatrix} (\xi\tau)_{\mathfrak{p}}(\xi\tau)_{\bar{\mathfrak{p}}}^{-1} & \\ & (\xi\tau)_{\mathfrak{p}}(\xi\tau)_{\bar{\mathfrak{p}}}^{-1} \end{pmatrix}$$

W/out $\mathcal{F}^{-1}\alpha_{\xi\tau}$-part, nebentypus is

$$\begin{pmatrix} (\xi\tau)_{\mathfrak{p}} & & & \\ & (\xi\tau)_{\mathfrak{p}} & & \\ & & (\xi\tau)_{\mathfrak{p}} & \\ & & & (\xi\tau)_{\bar{\mathfrak{p}}}^{-1} \end{pmatrix} \times \begin{pmatrix} (\xi\tau)_{\bar{\mathfrak{p}}}^{-1} & \\ & (\xi\tau)_{\bar{\mathfrak{p}}}^{-1} \end{pmatrix}$$

Right translation by $\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} \times \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$

$$\rightsquigarrow (\mathcal{F}^{-1}\alpha_{\tau\xi})\left(\begin{pmatrix} a_1^{-1} & & \\ & a_2^{-1} & \\ & & a_3^{-1} \end{pmatrix} C^{-1}D \begin{pmatrix} a_4 & & \\ & b_1 & \\ & & b_2 \end{pmatrix}\right)$$



$\mathcal{F}^{-1}\alpha_{\xi\tau}$-part contributes

$$\begin{pmatrix} (\xi\tau)_{\mathfrak{p}}^{-1} & & & \\ & (\xi\tau)_{\mathfrak{p}}^{-1} & & \\ & & \mathrm{triv} & \\ & & & \mathrm{triv} \end{pmatrix} \times \begin{pmatrix} (\xi\tau)_{\mathfrak{p}} & \\ & (\xi\tau)_{\mathfrak{p}} \end{pmatrix}$$

The const terms

$$\begin{array}{ccc}
\mathrm{Meas}(\Gamma_K, V_{GU(3,1),SO}) & \xrightarrow{\oplus \Phi_{g^p}} & \bigoplus_{g^p \in C^p} \mathrm{Meas}(\Gamma_K, M_{GU(2)}) \\
\rotatebox{90}{$\in$} & & \\
\mathbb{E}_\varphi^{\mathrm{Kling}} & &
\end{array}$$

$$\left(\Phi_{g^p}\left(\mathbb{E}_\varphi^{\mathrm{Kling}}\right)(\tau) \right)(h) = \int E^{\mathrm{Kling}}_{\varphi, \xi\tau} \left(\begin{pmatrix} 1 & & \\ & I_2 & \sigma \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & h & \\ & & \nu(h) \end{pmatrix} g^p w_p \right) d\sigma$$

$$\begin{array}{ccc}
GU(3,1)(\mathbb{Q}_p) & \longrightarrow & GL_4(\mathbb{Q}_p) \\
\rotatebox{90}{$\in$} & & \\
w_p & \longmapsto & \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix}
\end{array}$$

Moeglin-Waldspurger

$$= F_{\varphi,\xi\tau}\left(\begin{pmatrix} 1 & & \\ & h & \\ & & \nu(h) \end{pmatrix} g^p w_p \right) + \left(M_P F_{\varphi,\xi\tau} \right)\left(\begin{pmatrix} 1 & & \\ & h & \\ & & \nu(h) \end{pmatrix} g^p w_p \right)$$

$\uparrow$

intertwining operater w.r.t.

$$P = \begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$$

section for inducing $\xi_0\tau_0 |\cdot|^{\frac{k-3}{2}} \boxtimes \pi$

archimedean component $\Rightarrow$ vanish

$\downarrow$

$$F_{\varphi,\xi\tau}(g) = \int_{[U(2)]} f_{\xi\tau}\left(s^{-1} \begin{pmatrix} g & \\ & h_1 t \end{pmatrix} s \right) \varphi(h_1 t)\, (\xi_0\tau_0)^{-1}(\det h_1 t)\, dh_1$$

$$\nu(t) = \nu(g)$$

$$h \longmapsto F_{\varphi, \xi\tau}\left(\begin{pmatrix}1 & & \\ & h & \\ & & \nu(h)\end{pmatrix} \underbrace{g^p w_p}_{g}\right)$$ pairing w/ $\varphi' \in \pi$

$$\rightsquigarrow \prod_v Z_v$$

$$Z_v = \int_{U(2)(\mathbb{Q}_v)} f_{\xi\tau, v}\left(s^{-1}\begin{pmatrix} g_v & \\ & h_1\end{pmatrix} s\right) \langle h_1\varphi_v, \varphi'_v\rangle \, dh_1$$

If everything is unramified at v, $g_v \in GU(3,1)(\mathbb{Z}_p)$

then Z_v = normalization factor $\cdot\, L_v\left(\frac{k-1}{2}, BC(\pi) \times \xi_0\tau_0\right)$

const term $\sim \mathcal{L}^{\Sigma}_{\pi,\kappa,\xi} \cdot \mathcal{L}^{\Sigma}_{\xi,\mathbb{Q}}$

At an interpolation pt τ, $\xi\tau$ has ∞-type $(0,k)$, $k \geq 6$

interpolates $L^{\Sigma}\left(\frac{k-1}{2}, BC(\pi) \times \xi_0\tau_0\right)$

$L^{\Sigma}(k-2, \xi_0\tau_0|_{\mathbb{A}_{\mathbb{Q}}^{\times}})$

$$T_{\pi,\kappa,\xi} = T_\pi(\epsilon_{cyc}^2)\Big|_{G_\kappa}(\xi^{-1}) \otimes \Lambda_\kappa(\Psi_\kappa^{-1})$$

Use the geometric convention, $\epsilon_{cyc} \longleftrightarrow |\cdot|$

$$\det \rho_\pi = \epsilon_{cyc}^{-1}$$

$$L(0, T_{\pi,\kappa,\xi}(\tau)) = L(0, BC(\pi)\,|\cdot|^{-\frac{1}{2}} \cdot |\cdot|^2 \cdot (\xi_0\tau_0)^{-1}|\cdot|^{-\frac{k}{2}})$$

$$= L\left(\frac{3-k}{2}, BC(\pi) \times \xi_0^{-1}\tau_0^{-1}\right)$$

$$\approx L\left(\frac{k-1}{2}, BC(\pi) \times \xi_0\tau_0\right)$$

Galois rep associated to $E_{\varphi,\xi\tau}^{kling}$

$$\begin{pmatrix} \rho_\pi \epsilon_{cyc}^{\frac{1}{2}} & * & \\ & \xi\tau\,\epsilon_{cyc}^{-\frac{3}{2}} & \\ & & (\xi\tau)^{-c}\,\epsilon_{cyc}^{\frac{3}{2}} \end{pmatrix}$$

$$Z_p = \int M_{Q_{GU(3,3)}} f_{(\xi\tau)^{-c},p}\left(s^{-1} \begin{pmatrix} w_p & \\ & h_1 \end{pmatrix} s\, \Upsilon_p \right) \langle h_1 \varphi_p, \varphi_p' \rangle \, dh_1$$

$$= \int M_{Q_{GU(2,2)}} f'_{(\xi\tau)^{-c},p}\left(s'^{-1} \begin{pmatrix} I_2 & \\ & h_1 \end{pmatrix} s'\, \Upsilon_p' \right) \langle h_1 \varphi_p, \varphi_p' \rangle \, dh_1$$

$$s' = \begin{pmatrix} I_2 & -\frac{\xi}{2} \\ -I_2 & -\frac{\xi}{2} \end{pmatrix}, \qquad \Upsilon_p' = P_p (s')^{-1}$$

$$I\left(-s-\tfrac{1}{2}, \xi_0\tau_0\right)\Big|_{s=\frac{k-3}{2}}$$

$$\Downarrow$$

$$f'_{(\xi\tau)^{-c},p}(g) = f_{(\xi\tau)^{-c},p}\left(\begin{array}{c|c} A_g & \begin{matrix} -1 \\ \\ \end{matrix} \quad B_g \\ \hline \begin{matrix} 1 \end{matrix} \quad C_g & D_g \end{array} \right) \qquad \begin{pmatrix} I_2 & 1 & \\ & & 1 \end{pmatrix}$$

$$= |\nu_g|^{-s+\frac{3}{2}} \left| \det C_g{}^t \bar{C}_g \right|^{s-\frac{1}{2}} \xi_0\tau_0(\nu_g \det C_g)$$

$$\cdot \left(\mathcal{F}^{-1}\alpha_{\xi\tau}\right) \begin{pmatrix} 0 & C_g^{-1} D_g \\ 0 & 0 \end{pmatrix}$$

Apply local fcn'l eq for doubling zeta integrals

(Lapid – Rallis)

$$Z_p = \gamma_p\left(-2s, (\xi_0\tau_0)^{-1}\big|_{\mathbb{Q}_p^\times}\right) \gamma_p\left(-s, BC(\pi) \times (\xi_0\tau_0)^{-c}\right)$$

$$\int_{U(2)(\mathbb{Q}_p)} f'_{(\xi\tau)^{-c},p}\left(s'^{-1} \begin{pmatrix} I_2 & \\ & h_1 \end{pmatrix} s' \gamma'_p \right) \langle h_1 \varphi_p, \varphi'_p \rangle \, dh_1$$

$$= \gamma_p\left(-2s, (\xi_0\tau_0)^{-1}\big|_{\mathbb{Q}_p^\times}\right) \gamma_p\left(-s, BC(\pi) \times (\xi_0\tau_0)^{-c}\right)$$

$$\int_{GL_2(\mathbb{Q}_p)} |\det h_1|^{s+\frac{3}{2}} (\xi_0\tau_0)_{\wp} (\det h_1)$$

$$\mathcal{F}^{-1}\alpha_{\xi\tau} \begin{pmatrix} 0 & h_1 \\ 0 & 0 \end{pmatrix} \langle h_1 \varphi_p, \varphi'_p \rangle \, dh_1$$

Godement - Jacquet local fcn'l eq

$$= \gamma_p\left(-2s, (\xi_0\tau_0)^{-1}\big|_{\mathbb{Q}_p^\times}\right) \gamma_p\left(-s, BC(\pi) \times (\xi_0\tau_0)^{-c}\right)$$

$$\gamma_{\wp}(s+1, BC(\pi) \times \xi_0\tau_0) \int_{GL_2(\mathbb{Q}_p)} \mathbb{1}_{GL_2(\mathbb{Z}_p)}(h_1) \mathbb{1}_{\mathbb{Z}_p^\times}(a) \, (\xi\tau)^{-1}_{\wp} (\det h_1)$$

$$|\det h_1|^{-s+\frac{1}{2}} (\xi_0\tau_0)_{\wp} (\det h_1) \, dh_1$$

$$h_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \gamma_p\left(-2s, (\xi_0\tau_0)^{-1}\big|_{\mathbb{Q}_p^\times}\right) \gamma_{\bar{\wp}}\left(-s, BC(\pi) \times (\xi_0\tau_0)^{-1}\right) \cdot \mathrm{vol}$$