The non-degenerate Fourier-Jacobi coeff's

$\beta \in \mathrm{Her}_1(\mathcal{K})_{>0} = \mathbb{Q}_{>0}$ (denoted by n in Xin's lecture)

β-th FJ coeff at $g \in GU(3,1)(\mathbb{A}_f)$: $V_{GU(3,1),n,m} \longrightarrow H^0(\mathcal{C}_{g,n}, \mathcal{L}(\beta) \otimes \mathbb{Z}/p^m\mathbb{Z})$

$$\begin{array}{c} \mathcal{C}_{g,n} \\ \downarrow \\ sh_{GU(2),K_{g,n}} \end{array} \qquad \text{torsor of ab scheme} \qquad (*)$$

$g_p = I_4$

$K_{g,n,p} = \left\{ g \in GU(2)(\mathbb{Z}_p) \,\middle|\, \rho_{\mathfrak{p}}(g) \equiv \begin{pmatrix} \times & \times \\ 0 & 1 \end{pmatrix} \mod p^n \right\}$

Let $V^{J,\beta}_{GU(2)} = \varprojlim_m \varinjlim_n H^0(\mathcal{C}_{g,n}, \mathcal{L}(\beta) \otimes \mathbb{Z}/p^m\mathbb{Z})$

$\mathrm{Meas}(\Gamma_{\mathcal{K}}, V_{GU(3,1)})$

$\Big\downarrow$ β-th FJ coeff at $g \in GU(3,1)(\mathbb{A}_f^p)$

$\mathrm{Meas}(\Gamma_{\mathcal{K}}, V^{J,\beta}_{GU(2)})$

$\Big\downarrow$ $\ell_{\theta_i^J} : V^{J,\beta}_{GU(2)} \longrightarrow V_{GU(2)}$

pairing w/ θ_i^J, a chosen Jacobi form

along the fibre of $(*)$

$\mathrm{Meas}(\Gamma_{\mathcal{K}}, V_{GU(2)})$

$\Big\downarrow$ construct $\mathbb{h} \in \mathrm{Meas}(\Gamma_{\mathcal{K}}, V_{GU(2),\mathrm{ord}})$

and convoluations w/ $\mathbb{h}$

$\mathrm{Meas}(\Gamma_{\mathcal{K}}, V_{GU(2)} \times V_{GU(2),\mathrm{ord}})$

$\Big\downarrow$ p-adic Pertersson inner product for

families on $GU(2)$ (Hsieh, Wan)

$\mathrm{Meas}(\Gamma_{\mathcal{K}}, \hat{\mathcal{O}}_L^{ur})$

$$\mathbb{E}_{\varphi}^{\mathrm{Kling}} \longmapsto \langle \ell_{\theta_i^J}(\mathbb{E}^{\mathrm{Kling}}_{\varphi,\beta}), \mathbb{h} \rangle$$

$E^{\mathrm{Sieg}}_{\xi\tau}$ on $GU(3,3) = GU\begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix}$

$\Big\downarrow$ β-th FJ coeff w.r.t. $\begin{pmatrix} 1 & & * & \\ & I_2 & & \\ & & 1 & \\ & & & I_2 \end{pmatrix}$

Jacobi form on $U(2,2) = U\begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$

fcn on $\begin{pmatrix} 1 & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & & 1 & & \\ & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \end{pmatrix} = N \rtimes U(2,2)$

$E^{\mathrm{Sieg}}_{U(2,2),\xi\tau} \cdot \Theta^{J}_{U(2,2)}$

$\Big\downarrow$ res to $(N' \rtimes U(2)) \times U(2)$

Jacobi form on $U(2)$ $\quad \left(E^{\mathrm{Sieg}}_{U(2,2),\xi\tau}\Big|_{U(2)\times U(2)}\right) \cdot (\theta_3^J \boxtimes \theta_2)$

$\boxtimes$ autom form on $U(2)$

$\Big\downarrow$ pair w/ θ_1^J along N'

autom form on $U(2) \times U(2)$ $\quad \left(E^{\mathrm{Sieg}}_{U(2,2),\xi\tau}\Big|_{U(2)\times U(2)}\right) \cdot (1 \boxtimes \theta_2)$

doubling method $\;\|$

$$\sum_{h_2} \frac{L\left(\frac{k-2}{2}, BC(\pi_{h_2}) \times \xi_0 \tau_0 \lambda^{-1}\right)}{\langle h_2, h_2 \rangle} h_2 \boxtimes h_2' \theta_2$$

$\Big\downarrow$ pair w/ φ on the second factor

autom form on $U(2)$ $\quad \displaystyle\sum_{h_2} \frac{L\left(\frac{k-2}{2}, BC(\pi_{h_2}) \times \xi_0 \tau_0 \lambda^{-1}\right)}{\langle h_2, h_2 \rangle} h_2 \langle h_2' \theta_2, \varphi \rangle$

$\Big\downarrow$ pair w/ $h_\tau = \mathbb{h}(\tau)$

scalar $\quad L\left(\frac{k-2}{2}, BC(\pi_h) \times \xi_0 \tau_0 \lambda^{-1}\right) \langle h\theta_2, \varphi \rangle$

$= \left\langle \ell_{\theta_1^J}\left(\mathbb{E}^{\mathrm{Kling}}_{\varphi,\beta}\right), \mathbb{h} \right\rangle(\tau) \;=\;$ Pick $\mathbb{h}$ to be a CM family (involving choosing two Hecke char's)

$$E^{Sieg}_{3\tau,\beta}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} E^{Sieg}_{3\tau,\beta}\left(\begin{pmatrix}1 & & \sigma & \\ & I_2 & & \\ & & 1 & \\ & & & I_2\end{pmatrix} g\right)\psi(-\beta\sigma)\,d\sigma$$

$$= \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \sum_{\gamma\in Q\backslash GU(3,3)(\mathbb{Q})} f_{3\tau}\left(\gamma\begin{pmatrix}1 & & \sigma & \\ & I_2 & & \\ & & 1 & \\ & & & I_2\end{pmatrix} g\right)\psi(-\beta\sigma)\,d\sigma$$

$$Q = \begin{array}{cc} \begin{array}{cc} 3 & 3 \end{array} & \\ \begin{pmatrix} \times & \times \\ 0 & \times \end{pmatrix} & \begin{array}{c} 3 \\ 3 \end{array} \end{array}, \qquad P = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 1 & 2 \end{array} & \\ \begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \\ & \times & \times & \times \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 1 \\ 2 \end{array} \end{array}$$

$$GU(3,3) = \cancel{QP} \sqcup Q\begin{pmatrix} & & 1 & \\ & I_2 & & \\ -1 & & & \\ & & & I_2\end{pmatrix}P$$

$$P\cap\begin{pmatrix} & & 1 & \\ & I_2 & & \\ -1 & & & \\ & & & I_2\end{pmatrix}^{-1} Q\begin{pmatrix} & & 1 & \\ & I_2 & & \\ -1 & & & \\ & & & I_2\end{pmatrix}\Bigg\backslash P$$

$$= \sum_{\substack{w\in\mathcal{K}^2 \\ x\in\mathbb{Q} \\ \gamma\in Q'\backslash U(2,2)(\mathbb{Q})}} \begin{pmatrix}1 & w & x & \\ & I_2 & & \\ & & 1 & \\ & & -{}^t\bar{w} & I_2\end{pmatrix}\begin{pmatrix}1 & & & \\ & A_\gamma & & B_\gamma \\ & & 1 & \\ & C_\gamma & & D_\gamma\end{pmatrix}$$

$$Q' = \begin{array}{cc} \begin{array}{cc} 2 & 2 \end{array} & \\ \begin{pmatrix} \times & \times \\ 0 & \times \end{pmatrix} & \begin{array}{c} 2 \\ 2 \end{array} \end{array}$$

$$= \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} \sum_{\substack{w\in\mathcal{K}^2 \\ x\in\mathbb{Q} \\ \gamma\in Q'\backslash U(2,2)(\mathbb{Q})}} f_{3\tau}\left(\begin{pmatrix} & & 1 & \\ & I_2 & & \\ -1 & & & \\ & & & I_2\end{pmatrix}\begin{pmatrix}1 & w & x+\sigma & \\ & I_2 & & \\ & & 1 & \\ & & -{}^t\bar{w} & I_2\end{pmatrix}\begin{pmatrix}1 & & & \\ & A_\gamma & & B_\gamma \\ & & 1 & \\ & C_\gamma & & D_\gamma\end{pmatrix} g\right)\psi(-\beta\sigma)\,d\sigma$$

$$= \sum_{\substack{w \in K^2 \\ \gamma \in Q' \backslash U(2,2)(\mathbb{Q})}} \int_{\mathbb{A}_\mathbb{Q}} f_{\xi\tau}\left(\begin{pmatrix} & & 1 & \\ & & & I_2 \\ -1 & & & \\ & I_2 & & \end{pmatrix} \begin{pmatrix} 1 & w & \sigma & \\ & I_2 & & \\ & & 1 & \\ & & -{}^t\bar{w} & I_2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & A_\gamma & & B_\gamma \\ & & 1 & \\ & C_\gamma & & D_\gamma \end{pmatrix} g \right) \psi(-\beta\sigma)\, d\sigma$$

$$u(x,y,b) = \left(\begin{array}{cc|cc} 1 & x & b + \frac{x{}^t\bar{y} - y{}^t\bar{x}}{2} & y \\ & I_2 & {}^t\bar{y} & \\ \hline & & 1 & \\ & & -{}^t\bar{x} & I_2 \end{array}\right) \in N(\mathbb{A})$$

$$g \in U(2,2)(\mathbb{A})$$

$$\iota(g) = \begin{pmatrix} 1 & & & \\ & A_g & & B_g \\ & & 1 & \\ & C_g & & D_g \end{pmatrix}$$

$$E^{Sieg}_{\xi\tau,\beta}\left(u(x,y,b)\, \iota(g) \right)$$

$$= \sum_{\substack{w \in K^2 \\ \gamma \in Q' \backslash U(2,2)(\mathbb{Q})}} \psi\left(\beta \left(b + \frac{x{}^t\bar{y} + y{}^t\bar{x}}{2} + w{}^t\bar{y} + y{}^t\bar{w}\right)\right) FJ_\beta(\gamma g, x+w)$$

w/ $FJ_\beta(g, x) = \prod_v FJ_{\beta,v}(g_v, x_v)$

$$FJ_{\beta,v}(g_v, x_v) = \int_{\mathbb{Q}_v} f_{\xi\tau,v}\left(\begin{pmatrix} & & 1 & \\ & & & I_2 \\ -1 & & & \\ & I_2 & & \end{pmatrix} \begin{pmatrix} 1 & x_v & \sigma & \\ & I_2 & & \\ & & 1 & \\ & & -{}^t\bar{x}_v & I_2 \end{pmatrix} \iota(g_v) \right) \psi_v(-\beta\sigma)\, d\sigma$$

$\omega_{\beta,v}$ Weil rep $\left(\begin{array}{l} \text{w.r.t. a fixed } \lambda : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C} \\ \text{s.t. } \lambda|_{\mathbb{A}_{\mathbb{Q}}^\times} = \chi_{\mathcal{K}/\mathbb{Q}} \end{array}\right)$

$$N \rtimes U(2,2) \curvearrowright \{\text{Schwartz fcn's on } \mathcal{K}_v^2\}$$

$u(x,y,b) \quad \iota(g)$

$$\psi\left(\beta\left(b + \frac{x\,{}^t\bar{y} + y\,{}^t\bar{x}}{2} + w\,{}^t\bar{y} + y\,{}^t\bar{w}\right)\right) FJ_{\beta,v}(g, w+x)$$

$$= \omega_{\beta,v}\left(u(x,y,b)\right) \underbrace{FJ_{\beta,v}(g,w)}_{\text{Schwartz fcn on } w \in \mathcal{K}_v^2}$$

$$FJ_{\beta,v}\left(\begin{bmatrix} A & B \\ 0 & {}^t\bar{A}^{-1} \end{bmatrix} g,\ w\right) = |\det A\,{}^t\bar{A}|_v^{\frac{k}{2}} (\xi_0\tau_0)_v(\det A)\ \psi_v(\beta w B\,{}^t\bar{A}\,{}^t\bar{w})$$

$$FJ_{\beta,v}(g, wA)$$

$$= |\det A\,{}^t\bar{A}|^{\frac{k-1}{2}} (\xi_0\tau_0\lambda^{-1})_v(\det A)$$

$$|\det A\,{}^t\bar{A}|_v^{\frac{1}{2}} \lambda_v(\det A)\ \psi_v(\beta w B\,{}^t\bar{A}\,{}^t\bar{w})\ FJ_{\beta,v}(g, wA)$$

$$\Rightarrow FJ_{\beta,v}(g,w) \approx f^{U(2,2)}_{\xi\tau}(g)\ \omega_{\beta,v}(g)\Phi_v(w)$$

$$\cap$$

$$I(s, \xi_0\tau_0\lambda^{-1})\Big|_{s=\frac{k-3}{2}}$$

$$E^{\text{sieg}}_{\xi\tau,\beta}\left(u(x,y,b)\,\iota(g)\right)$$

$$\approx \sum_{\substack{\gamma \in Q_2 \backslash U(2,2)(\mathbb{Q}) \\ w \in \mathcal{K}^2}} \omega_{\beta, v}\Big(u(x,y,b)\, \iota(g) \Big)\, \Phi(w)\, f_{3\tau}^{U(2,2)}(\gamma g)$$

$$= E^{Sieg}_{U(2,2), 3\tau}(g)\ \Theta^{J}_{U(2,2)}\Big(u(x,y,b)\, \iota(g) \Big)$$