Lectures on Iwasawa theory for Automorphic Forms

Xin Wan

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Main Results

We first state our main result, leaving the background and motivation to the next lecture of Francesc.

- Let p be an odd prime and f be a cuspidal eigenform of weight 2, trivial character and conductor N coprime to p. Let T_f be the Galois representation associated to the automorphic representation π of f over a coefficient ring $\mathscr{O}_L$ finite over $\mathbb{Z}_p$. The Hodge-Tate weight of T_f is $(0,1)$.
- Let $\mathscr{K}$ be a quadratic imaginary field such that p is split. We fix an isomorphism $\iota : \mathbb{C} \simeq \mathbb{C}_p$, which determines a prime $\mathfrak{p}$ of $\mathscr{K}$ above p. Let ξ be an algebraic Hecke character of $\mathscr{K}$.

Let $\mathscr{K}_\infty$ be the $\mathbb{Z}_p^2$-extension of $\mathscr{K}$ and let $\Gamma_{\mathscr{K}} \simeq \mathbb{Z}_p^2$ be its Galois group over $\mathscr{K}$. Define the Iwasawa algebra $\Lambda_{\mathscr{K}} := \mathscr{O}_L[[\Gamma_{\mathscr{K}}]]$. Let $\Gamma_{\mathscr{K}}^{\pm} \simeq \mathbb{Z}_p$ be the subspace on which the complex conjugation acts by ± 1. Let

$$\Psi_{\mathscr{K}} : G_{\mathscr{K}} \to \Gamma_{\mathscr{K}} \hookrightarrow \mathscr{O}_L[[\Gamma_{\mathscr{K}}]]^{\times}.$$

We define $T_{\pi,\mathscr{K},\xi}$ for the Galois representation

$$T_f(\xi^{-1}) \otimes \epsilon_{\mathrm{cyc}}^2 \otimes \Lambda_{\mathscr{K}}(\Psi_{\mathscr{K}}^{-1}).$$

- We define the Selmer group

$$\mathrm{Sel}_{\pi,\mathscr{K},\xi} := \ker\{H^1(\mathscr{K}, T_{\pi,\mathscr{K},\xi} \otimes_{\Lambda_{\mathscr{K}}} \Lambda^*_{\mathscr{K}}) \to \prod_{v \nmid \mathfrak{p}} H^1(\mathscr{K}, T_{\pi,\mathscr{K},\xi} \otimes_{\Lambda_{\mathscr{K}}} \Lambda^*_{\mathscr{K}})\}.$$

  Let $X_{\pi,\mathscr{K},\xi}$ be the Pontraygin dual of $\mathrm{Sel}_{\pi,\mathscr{K},\xi}$. Let Σ be a finite set of primes we can also define $X^{\Sigma}_{\pi,\mathscr{K},\xi}$ by relaxing the local Selmer conditions at places in Σ.
- We note that this kind of Selmer condition is different from the Bloch-Kato condition at p-adic places: it is empty at one place above p and is the full space at the other place. This is irrelevant the reduction type at p (ordinary or non-ordinary).

- On the analytic side there is a p-adic L-function $\mathscr{L}_{\pi,\mathscr{K},\xi} \in \Lambda_{\mathscr{K}}$ interpolating critical values of the Rankin-Selberg product of f with CM forms associated to CM characters which are specializations of $\xi\Psi_{\mathscr{K}}$ with weight higher than 2 (the weight of f) (More details in Zheng's lectures). We can also define Σ-imprimitive p-adic L-functions $\mathscr{L}^{\Sigma}_{\pi,\mathscr{K},\xi}$ by removing Euler factors at primes at Σ.

The corresponding Iwasawa main conjecture is the following.

Conjecture

The characteristic ideal $\mathrm{char}(X^{\Sigma}_{\pi,\mathscr{K},\xi})$ *is generated by the* $(\mathscr{L}^{\Sigma}_{\pi,\mathscr{K},\xi})$.

Our main result is the following: suppose Σ contains all bad primes (i.e. primes where $\mathscr{K}/\mathbb{Q}$, π or ξ is ramified).

- Suppose N is square-free.
- The mod p Galois representation $\bar{T}_f$ is absolutely irreducible over $G_{\mathscr{K}}$.
- There is a prime q nonsplit in $\mathscr{K}$ where π is ramified. If 2 is nonsplit in $\mathscr{K}$ then we assume 2 is ramified at $\mathscr{K}$.

Then we have one side inclusion

$$\mathrm{char}_{\Lambda_{\mathscr{K}}}(X^{\Sigma}_{\pi,\mathscr{K},\xi}) \subseteq (\mathscr{L}^{\Sigma}_{\pi,\mathscr{K},\xi})$$

up to height one primes of $\mathscr{O}_L[[\Gamma^+_{\mathscr{K}}]]$.

In applications we often combine this with results of vanishing of μ-invariant (say of Burungale) to get rid of the height one primes of $\mathscr{O}_L[[\Gamma^+_{\mathcal{K}}]]$ and get full equality.
This is a special case of Iwasawa theory under Panchishkin condition in general.

- Suppose T is a geometric $\mathbb{Z}_p$-Galois representation of $G_{\mathbb{Q}}$ and $V := T \otimes \mathbb{Q}_p$. Then we have the Hodge-Tate decomposition

$$V \otimes \mathbb{C}_p = \oplus_i \mathbb{C}_p(i)^{h_i}$$

  where $\mathbb{C}_p(i)$ is the i-th Tate twist and h_i is the multiplicity.
- Let d be the dimension of T and let $d^{\pm}$ be the dimensions of the ± 1-eigenvalue subspaces of the complex conjugation.

Following Greenberg we assume:

- $d^+ = \sum_{i>0} h_i$ (the p-adic criticality condition).
- There is a d^+-dimensional $\mathbb{Q}_p$-subspace V^+ of V which is stable under the action of the decomposition group $G_{\mathbb{Q}_p}$ at p such that $V^+ \otimes \mathbb{C}_p = \oplus_{i>0} \mathbb{C}_p(i)^{h_i}$ (the Panchishkin condition).

Write $T^+ := V^+ \cap T$. Greenberg defined the following local Selmer condition

$$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V/T) = \mathrm{Ker}\{H^1(\mathbb{Q}_p, V/T) \to H^1(\mathbb{Q}_p, \frac{V/T}{V^+/T^+})\}.$$

In other words under the Panchishkin condition the local Selmer condition above at p is very analogous to the ordinary case.

- A typical example is the following. Let f be a cuspidal eigenform of weight k and g be a CM form of weight k' with respect to a quadratic imaginary field $\mathscr{K}$ such that p splits. Then g is nearly ordinary at p by definition. Assume $k + k'$ is an odd number. We consider the Rankin-Selberg product of f and g and see if it satisfies the Panchishkin condition.
  1. If $k > k'$, then the Panchishkin condition is true if f is ordinary;
  2. If $k' > k$, then the Panchishkin condition is always true. This is precisely the context of our results.

We next discuss the proof. The main idea is to use congruences between Eisenstein series and cusp forms on the unitary group $\mathrm{U}(3,1)$.

We first define various unitary groups. Let $\delta' \in \mathscr{K}$ be a totally imaginary element such that $-i\delta'$ is positive. Let $d = \mathrm{Nm}(\delta')$ which we assume to be a p-adic unit. Let $\mathrm{U}(2) = \mathrm{U}(2,0)$ (resp. $\mathrm{GU}(2) = \mathrm{GU}(2,0)$) be the unitary group (resp. unitary similitude group) associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} \mathfrak{s}\delta' & \\ & \delta' \end{pmatrix}$ for some $\mathfrak{s} \in \mathbb{Z}_+$ prime to p. More precisely $\mathrm{GU}(2)$ is the group scheme over $\mathbb{Z}$ defined by: for any $\mathbb{Z}$ algebra A,

$$\mathrm{GU}(2)(A) = \{g \in \mathrm{GL}_2(A \otimes_{\mathbb{Z}} \mathscr{O}_{\mathscr{K}}) | {}^t\bar{g}\zeta g = \lambda(g)\zeta,\ \lambda(g) \in A^\times.\}$$

The map $\lambda : \mathrm{GU}(2) \to \mathbb{G}_m$, $g \mapsto \lambda(g)$ is called the similitude character and $\mathrm{U}(2) \subseteq \mathrm{GU}(2)$ is the kernel of λ.

The group $\mathrm{GU}(2)$ is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathscr{K}) | g\zeta {}^t\bar{g} = \det(g)\zeta\}.$$

Then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariant $\mathrm{inv}_v(D) = (-\mathfrak{s}, -D_{\mathscr{K}/\mathbb{Q}})_v$ (the Hilbert symbol).
Let $G = \mathrm{GU}(3,1)$ (resp. $\mathrm{U}(3,1)$) be the similarly defined unitary similitude group (resp. unitary group) over $\mathbb{Z}$ associated to the skew-Hermitian matrix $\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}$. We denote this Hermitian space as V. Let $P \subseteq G$ be the parabolic subgroup of $\mathrm{GU}(3,1)$ consisting of those matrices in G of the form $\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$.

Let N_P be the unipotent radical of P. Then if $X_{\mathscr{K}}$ is the 1-dimensional space over $\mathscr{K}$,

$$M_P := \mathrm{GL}(X_{\mathscr{K}}) \times \mathrm{GU}(2) \hookrightarrow \mathrm{GU}(V),\ (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

is the Levi subgroup of P. Let $G_P := \mathrm{GU}(2) \subseteq M_P$ be the set of elements $\mathrm{diag}(1, g_1, \mu(g_1))$ as above. Let δ_P be the modulus character for P. We usually use a more convenient character δ such that $\delta^3 = \delta_P$.

Let $(r, s) = (3, 3)$ or $(2, 2)$ or $(3, 1)$ or $(2, 0)$. Then the unbounded Hermitian symmetric domain for $\mathrm{GU}(r, s)$ is

$$X^+ = X_{r,s} = \{\tau = \begin{pmatrix} x \\ y \end{pmatrix} | x \in M_s(\mathbb{C}), y \in M_{(r-s)\times s}(\mathbb{C}),$$
$$i(x^* - x) > iy^*\zeta^{-1}y\}.$$

We use x_0 to denote the Hermitian symmetric domain for $\mathrm{GU}(2)$, which is just a point.

Let $G_{r,s} = \mathrm{GU}(r,s)$ and $H_{r,s} = \mathrm{GL}_r \times \mathrm{GL}_s$. Let $G_{r,s}(\mathbb{R})^+$ be the subgroup of elements of $G_{r,s}(\mathbb{R})$ whose similitude factors are positive. If $s \neq 0$ we define a co-cycle:

$$J : G_{r,s}(\mathbb{R})^+ \times X^+ \to H_{r,s}(\mathbb{C})$$

by $J(\alpha,\tau) = (\kappa(\alpha,\tau), \mu(\alpha,\tau))$, where for $\tau = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$ (block matrix with respect to the partition $(s + (r-s) + s)$), where

$$\kappa(\alpha,\tau)=\begin{pmatrix}\bar{h}^t x+\bar{d} & \bar{h}^t y+l\bar{\zeta}\\ -\bar{\zeta}^{-1}(\bar{g}^t x+\bar{f}) & -\bar{\zeta}^{-1}\bar{g}^t y+\bar{\zeta}^{-1}\bar{e}\bar{\zeta}\end{pmatrix},\ \mu(\alpha,\tau)=hx+ly+d$$

in the $\mathrm{GU}(3,1)$ case and

$$\kappa(\alpha,\tau)=\bar{h}^t x+\bar{d},\ \mu(\alpha,\tau)=hx+d$$

in the $\mathrm{GU}(3,3)$ case.

Let $i \in X^+$ be the point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$. Let K_∞^+ be the compact subgroup of $\mathrm{U}(r,s)(\mathbb{R})$ stabilizing i and let K_∞ be the group generated by K_∞^+ and $\mathrm{diag}(1_{r+s}, -1_s)$. Then

$$K_\infty^+ \to H(\mathbb{C}),\ k_\infty \mapsto J(k_\infty, i)$$

defines an algebraic representation of K_∞^+.

Definition

A weight $\underline{k}$ is defined to be an $(r+s)$-tuple

$$\underline{k} = (a_1, \cdots, a_j; b_1, ..., b_s) \in \mathbb{Z}^{r+s}$$

with $a_1 \geq \cdots a_r \geq -b_1 - \cdots - b_s$, $a_r + b_1 \geq r + s$.

We define the representation $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$. We also note that if each $\underline{k} = (0, ..., 0; \kappa, ..., \kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one-dimensional.
For a weight $\underline{k}$, define $\|\underline{k}\|$ by:

$$\|\underline{k}\| := a_1 + \cdots a_r + b_1 + \cdots + b_s.$$

Definition

Let U be an open compact subgroup of $G(\mathbb{A}_f)$. We denote by $M_{\underline{k}}(U, \mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_f)$ such that for $\tau \in X^+$, $\alpha \in G(\mathbb{Q})^+$ and $u \in U$ we have

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|}\rho^{\underline{k}}(J(\alpha, \tau))f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F : G(\mathbb{A}) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(\mathbb{Q}) \times K_\infty^+ \times U \times Z(\mathbb{A})$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, i)^{-1})F(g)\chi^{-1}(z).$$

We can associate a $L_{\underline{k}}(\mathbb{C})$-valued function on $X^+ \times G(\mathbb{A}_f)/U$ given by

$$f(\tau, g) := \chi_f(\mu(g))\rho^{\underline{k}}(J(g_\infty, i))F((f_\infty, g)) \qquad (1.1)$$

where $g_\infty \in G(\mathbb{R})$ such that $g_\infty(i) = \tau$. If this function is holomorphic, then we say that the automorphic form F is holomorphic.

We consider the Jacquet-Langlands correspondence of π to the quaternion algebra D associated to our unitary group $\mathrm{U}(2)$, still denote as π by abuse of notation. Let ψ be a Hecke character of $\mathbb{A}_{\mathscr{K}}$ whose restriction to $\mathbb{A}_{\mathbb{Q}}$ is trivial. Then naturally one can associate from it an automorphic representation π_ψ of $\mathrm{GU}(2)$ with central character ψ. These determine a representation $\rho := \pi_\psi \times \xi$ of $M_P \simeq \mathrm{GU}(2) \times \mathrm{Res}_{\mathscr{K}/\mathbb{Q}}\mathbb{G}_m$. Here for $g \in \mathrm{GU}(2)$ and $x \in \mathbb{A}_{\mathscr{K}}^\times$, we identify with it an element

$$m(g,x) = \begin{pmatrix} \mu(g)x^{-1} & & \\ & g & \\ & & x \end{pmatrix} \in M_P(\mathbb{R}).$$

We consider the induced representation (both locally and globally) $I_P^{\mathrm{GU}(3,1)}(\rho)$. For each section F in it we can associate an

$$F_z(g) := \delta(m)^{\frac{3}{2}+z}\rho(m)f(k), g = mnk \in P(\mathbb{R})K_\infty.$$

For F being a global section we can form a Klingen Eisenstein series

$$E(F, z, g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} F_z(\gamma g).$$

General theory of Eisenstein series gives that it is absolutely convergent when $\mathrm{Re}(z) >> 0$ and has meromorphic continuation to the whole complex plane for z.

- We need another kind of Eisenstein series on $\mathrm{GU}(n, n)$, namely the Siegel Eisenstein series. The Siegel Eisenstein series are much easier than Klingen Eisenstein series to compute – its Fourier-expansion is already computed by Shimura. On the other hand, the Klingen Eisenstein series can be realized from Siegel Eisenstein series (the doubling method, see later).
- We let $\mathrm{GU}(n, n)$ be the unitary similitude group associated to the matrices $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.

- Let $Q = Q_n$ be the Siegel parabolic subgroup of G_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$.
- For a place v of $\mathbb{Q}$ and a character χ of $\mathscr{K}_v^\times$ we let $I_n(\chi_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathbb{Z}_v)$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi_v(\det D_q)f(k)$ for all $q \in Q_n(\mathbb{Q}_v) \cap K_{n,v}$ (we write q as block matrix $q = \begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$).
- For $z \in \mathbb{C}$ and $f \in I(\chi)$ we also define a function $f(z, -) : G_n(\mathbb{Q}_v) \to \mathbb{C}$ by $f(z, qk) := \chi(\det D_q))|\det A_q D_q^{-1}|_v^{z+n/2} f(k), q \in Q_n(\mathbb{Q}_v)$ and $k \in K_{n,v}$.

Globally for a Hecke character $\chi = \otimes\chi_v$ of $\mathbb{A}_{\mathcal{K}}^{\times}$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{sph} \in I_n(\chi_v)$ (invariant under $K_{n,v}$) such that $f_v^{sph}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified.
For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of

$$\{\mathrm{Re}(z) > n/2\} \times G_n(\mathbb{A}_{\mathbb{Q}}).$$

The defined automorphic form is called Siegel-Eisenstein series.

We have the following theorem about associating Galois representations to automorphic forms on unitary groups.This is accomplished by Harris' book project (from work of many people including Harris, Taylor, Shin, Morel, etc). Let π be an irreducible automorphic representation of $\mathrm{GU}(r, s)(\mathbb{A}_{\mathbb{Q}})$ generated by a holomorphic cuspidal eigenform with weight $\underline{k} = (a_1, \cdots, a_r; b_1, \cdots, b_s)$ and central character χ_π. Let $\Sigma(\pi)$ be a finite set of primes of $\mathbb{Q}$ containing all the primes at which π is unramified and all the primes dividing p.

There is a Galois representation $R_p(\pi) : G_{\mathcal{K}} \to \mathrm{GL}_n(\mathcal{O}_L)$ satisfying

- $R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi)$, and for such a place $w \nmid p$:

$$\det(1 - R_p(\pi)(\mathrm{Frob}_w)q_w^{-s}) = L(BC(\pi)_w \otimes \chi^c_{\pi,w}, s + \frac{1-n}{2})^{-1}$$

Here, the Frob_w is the geometric Frobenius and BC means the base change from $\mathrm{U}(r,s)$ to GL_{r+s}.

Suppose π_v is semi-ordinary (to be defined in Zheng's lectures) with respect to $\underline{k}$ and unramified at all primes v dividing p. Then we have the following description when restricting to $G_{\mathfrak{p}}$:

$$R_p(\pi)|_{G_{\mathfrak{p}}} \simeq \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & \xi_{2,v}\epsilon^{-\kappa_2} & * \\ & & & \xi_{1,v}\epsilon^{-\kappa_1} \end{pmatrix} \tag{1.2}$$

where $\xi_{1,v}$ and $\xi_{2,v}$ are unramified characters and also

$$R_p(\pi)|_{G_{\bar{\mathfrak{p}}}} \simeq \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}.$$

This is a standard fact from the local-global compatibility at p.

Main Strategy

- Construct a p-adic family of Klingen Eisenstein series on $\mathrm{U}(3,1)$ so that the constant terms are divisible by $\mathscr{L}_{\pi,\mathscr{K},\xi}\cdot\mathscr{L}_{\xi'}$, where $\mathscr{L}_{\xi'}$ is the Dirichlet p-adic L-function associated to $\xi' := \xi|_{\mathbb{A}_{\mathbb{Q}}^{\times}}$.
- Prove that the Klingen Eisenstein family is co-prime to $\mathscr{L}_{\pi,\mathscr{K},\xi}$. This is the main technical difficulty (more details in Zheng's lectures). We can show that, using non-cuspidal semi-ordinary Hida theory (see Francesc and Zheng's lectures), these two steps imply the Hecke eigenvalues of the Klingen Eisenstein family is congruent to Hecke eigenvalues of the cuspidal family modulo $\mathscr{L}_{\pi,\mathscr{K},\xi}$.

Remark

The use of semi-ordinary family is a key ingredient for the whole argument — when restricting to an appropriate subspace of the weight space, it is a Hida theory over the Iwasawa algebra instead of a Coleman-Mazur theory, which can be only defined over a small affinoid ring. This is crucial for doing Iwasawa theory. Such phenomenon was first observed, in the context of arithmetic group cohomolog by Hida in 1995, and then later on studied by Tilouine-Urban for symplectic groups. Here we develop the theory in the context of coherent cohomology.

- Pass to Galois side, the above information implies we have congruences for Galois representations for Klingen Eisenstein family and cuspidal family modulo $\mathscr{L}_{\pi,\mathscr{K},\xi}$. The Galois representation associated to the Klingen Eisenstein family has the form $\mathrm{diag}(\chi', \rho_\pi, \chi)$ where χ' and χ are two characters. The Galois representation for cuspidal family is more irreducible – in fact at most two irreducible components.
- From the above congruences one constructs enough elements in the Selmer groups from the congruences between Galois associated to the Eisenstein family and cuspidal family, proving the desired bound. This is called "lattice construction" (generalized Ribet's lemma).

We now focus on the first and second step. The doubling method (by Piatetski-Shapiro, generalized by Garrett and written in Shimura's book) is a key ingredient for this. The idea for step 1 is first construct a family of Siegel Eisenstein series, and then pullback to get a family of Klingen Eisenstein series. We now explain the pullback formula. We define $\mathrm{GU}(3,3)'$ to be the unitary similitude group associated to:

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$$

and $\mathrm{GU}(2,2)'$ to be the unitary group associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}.$$

We define embeddings

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(3,3)'$$

and

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(2,0) \times \mathrm{GU}(0,2), \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(2,2)'$$

by $\alpha(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$ and $\alpha'(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$.

We also define isomorphisms:

$$\beta : \mathrm{GU}(3,3)' \xrightarrow{\sim} \mathrm{GU}(3,3)$$

and

$$\beta' : \mathrm{GU}(2,2)' \xrightarrow{\sim} \mathrm{GU}(2,2)$$

by

$$g \mapsto S^{-1} g S$$

or

$$g \mapsto S'^{-1} g S'$$

Here

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & -\frac{\zeta}{2} \\ & & 1 & \\ & -1 & & -\frac{\zeta}{2} \end{pmatrix}$$

and

$$S' = \begin{pmatrix} 1 & -\frac{\zeta}{2} \\ -1 & -\frac{\zeta}{2} \end{pmatrix}$$

We write γ and γ' for the embeddings $\beta \circ \alpha$ and $\beta' \circ \alpha'$, respectively.

Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$. Given a cuspform φ on $\mathrm{GU}(2)$ we consider

$$F_\varphi(f;z,g) := \int_{\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} f(z, S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1 g)\varphi(g_1 h)dg_1,$$

$$f \in I_3(\chi), g \in \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q}), h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), \mu(g) = \mu(h)$$

or

$$F'_\varphi(f';z,g) = \int_{\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} f'(z, S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1 g)\varphi(g_1 h)dg_1$$

$$f' \in I_2(\chi), g \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), \mu(g) = \mu(h)$$

This is independent of h. We see that the above integrals can be factorized as local integrals, which we denote as $F_{\varphi_v}(f_v; z, g_v)$ and $F'_{\varphi_v}(f'_v; z, g_v)$, respectively. The pullback formulas are the identities in the following proposition.

Proposition

Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^{\times}$.
(i) If $f' \in I_2(\chi)$, then $F'_{\varphi}(f'; z, g)$ converges absolutely and uniformly for (z, g) in compact subsets of $\{\mathrm{Re}(z) > 1\} \times \mathrm{GU}(2,0)(\mathbb{A}_{\mathbb{Q}})$, and for any $h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h) = \mu(g)$

$$\begin{aligned} &\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f'; z, S'^{-1}\alpha(g, g_1 h)S')\bar{\chi}(\det g_1 h)\varphi(g_1 h)dg_1 \\ = \;& F'_{\varphi}(f'; z, g). \end{aligned}$$

Proposition

(continued) (ii) If $f \in I_3(\chi)$, then $F_\varphi(f; z, g)$ converges absolutely and uniformly for (z, g) in compact subsets of $\{Re(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q})$ such that $\mu(h) = \mu(g)$

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} E(f; z, S^{-1}\alpha(g, g_1 h)S)\bar{\chi}(\det g_1 h)\varphi(g_1 h) dg_1$$
$$= \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{GU}(3,1)(\mathbb{Q})} F_\varphi(f; z, \gamma g),$$

with the series converging absolutely and uniformly for (z, g) in compact subsets of

$$\{\mathrm{Re}(z) > 3/2\} \times \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q}).$$

We need to make appropriate choices of local Siegel sections (most importantly the section at the p-adic place), so that

- It interpolates in p-adic analytic families in terms of formal Fourier expansion. A certain Siegel-Weil section enables us to take care of the q-expansion.
- It pulls back to the semi-ordinary Klingen Eisenstein section. A computation says that it is the section supported in the "small cell", which is hard to compute directly (the "big cell" sections are easier to compute). An idea from Skinner-Urban is to use a comparison between the global functional equations of Siegel and Klingen Eisenstein series.

We first briefly summarize the strategy of Skinner-Urban, which uses Eisenstein congruences on $\mathrm{U}(2,2)$.

- Write out the Klingen Eisenstein series as

$$E_{\mathrm{Kling}} = \sum_T a_T q^T$$

where T runs over 2×2 Hermitian matrices.

- From the doubling method, we have

$$a_T = \langle \mathrm{FJ}_T E_{\mathrm{Sieg}}, \varphi \rangle_{\mathrm{U}(1,1)}.$$

- Some linear combination of $\mathrm{FJ}_T E_{\mathrm{Sieg}}$ is computed as a product of Eisenstein series and theta functions on $\mathrm{U}(1,1)$.
- They choose an appropriate CM eigenform, and apply certain Hecke operators to pick up its eigen-component in the theta function. Then the corresponding linear combination of a_T's are Rankin-Selberg products of f with this CM forms.

Unlike $\mathrm{U}(2,2)$, the group $\mathrm{U}(3,1)$ is not quasi-split, and thus we only have Fourier-Jacobi expansion at a cusp $[g]$ ($g \in \mathrm{U}(3,1)(\mathbb{A}_f)$)

$$\mathrm{FJ}(F) = \sum_{n \in \mathbb{Q}} a_n q^n,$$

where $a_n \in H^0(\mathscr{Z}_{[g]}, \mathscr{L}(n))$ where $\mathscr{Z}_{[g]}$ is a boundary component (two dimensional Abelian varieties) and $\mathscr{L}(n)$ is a line bundle on it. The a_n is regarded as a form on the Klingen parabolic group $P = N \cdot \mathrm{U}(2)$ modulo its center, where N consists of matrices of the form $\begin{pmatrix} 1 & \times & \times & \times \\ & 1 & & \times \\ & & 1 & \times \\ & & & 1 \end{pmatrix}$.

- The $H^0(\mathscr{Z}_{[g]}, \mathscr{L}(n))$ is a finite dimensional space of theta functions. We can define a functional $l_{\theta,n}$ (on the space of forms on $\mathrm{U}(3,1)$) to be the n-th Fourier-Jacobi coefficient composed with pairing against a theta function θ (which we concretely write down). Our goal is to study p-adic properties of $l_{\theta,n}$ applied to the Klingen Eisenstein family.
- By the pullback formula, we first computed the n-th Fourier-Jacobi coefficient of the Siegel Eisenstein series E_{Sieg}, regarded as a form on the group $N' \cdot \mathrm{U}(2,2)$.

Here the N' consists of matrices of the form

$$\begin{pmatrix} 1 & \times & \times & \times & \times & \times \\ & 1 & & \times & & \\ & & 1 & \times & & \\ & & & 1 & & \\ & & & \times & 1 & \\ & & & \times & & 1 \end{pmatrix}.$$

The n-th Fourier-Jacobi coefficient is essentially a product of Siegel Eisenstein series $E_{\mathrm{U}(2,2)}$ on $\mathrm{U}(2,2)$ and a theta function Θ on the Jacobi group $N' \cdot \mathrm{U}(2,2)$.

In Step 2 we restrict this to the subgroup

$$(N \cdot \mathrm{U}(2)) \times \mathrm{U}(2) \subset (\mathrm{U}(3,1) \times \mathrm{U}(2)) \cap (N' \cdot \mathrm{U}(2,2)).$$

Another computation shows that Θ essentially restricts to a form $\theta_4 \times \theta_2$ on $(N \cdot \mathrm{U}(2)) \times \mathrm{U}(2)$. Applying a functional $l_{\theta^\star}$ (which is pairing with a *fixed* theta function $\theta^\star$ to the θ_4-component above, given by integration the product of $\theta^\star$ and θ_4 over N), we show that $\langle \theta_4, \theta^\star \rangle_N$ is a constant function on $\mathrm{U}(2)$, and then end up with a theta function θ_2 on (the lower) $\mathrm{U}(2)$.

In sum we have

$$l_{\theta^\star}(a_n(E_{Kling})) = \langle E_{\mathrm{U}(2,2)}|_{\mathrm{U}(2)\times\mathrm{U}(2)}, \varphi\cdot\theta_2\rangle_{1\times\mathrm{U}(2)}$$

regarded as a form on the first $\mathrm{U}(2)$, which is the $\mathrm{U}(2)$ in the Levi of the Klingen parabolic subgroup $P\subset\mathrm{GU}(3,1)$.

In Step 3, to study the p-adic properties, we study the pairing of it against another family of forms h on $\mathrm{U}(2)$, which we construct via theta lifts from $\mathrm{U}(1)$ to $\mathrm{U}(2)$. So we obtain

$$\langle\langle E_{\mathrm{U}(2,2)}|_{\mathrm{U}(2)\times\mathrm{U}(2)}, \varphi\cdot\theta_2\rangle_{1\times\mathrm{U}(2)}, h\rangle_{\mathrm{U}(2)} = (*)\cdot\langle h, \varphi\cdot\theta_2\rangle.$$

To obtain this formula, we use the doubling method formula for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ applied to h. The $(*)$ is some p-adic L-function factor for h coming from this. A technique of Hsieh implies that this pairing can be interpolated in p-adic families.

- We use Ichino's triple product formula (i.e the L-function for the tensor product $\pi_h \otimes \pi_{\theta_2} \otimes \pi_f$) to evaluate this:

$$(\int_{[U(2)]} h(g)\theta_2(g)fdg)(\int_{[U(2)]} \tilde{h}(g)\tilde{\theta}_2(g)\tilde{f}(g)dg)$$

  where the ˜ means forms in the dual space of the corresponding automorphic representations.
- As we choose h to be a CM form. The triple product splits into two Rankin-Selberg p-adic L-functions of π with two CM characters, one with weight higher than the weight of f and one with weight lower than that of f.

- We can make choice of the Hecke characters corresponding to h and θ so that the p-adic L-functions involved above are units in the Iwasawa algebra times a nonzero constant (this involves work of Hida and Hsieh). This means up to powers of p, some linear combination of the Fourier-Jacobi functional is co-prime to the p-adic L-function we study, completing step (2) of the argument.
- On the other hand, non-cuspidal Hida theory gives a semi-ordinary Hida family of cusp forms $\mathbf{H}$ which is congruent to E_{Kling} modulo $\mathscr{L}_{\pi,\mathscr{K},\xi}\cdot\mathscr{L}_{\xi'}$.

Eisenstein Ideal

Definition

Let $\mathbb{T}$ be the reduced Hecke algebra on the semi-ordinary cuspidal families on $\mathrm{U}(3,1)$ with the associated pseudo Galois representation $\rho_{\mathbb{T}}$. We define the ideal I of $\mathbb{T}$ to be generated by $\{t-\lambda(t)\}_t$ for t's in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of t on $\mathbf{E}_{\mathrm{Kling}}$. Then it is easy to see that the structure map $\Lambda_{\mathscr{K}} \to \mathbb{T}/I$ is surjective. Suppose the inverse image of I in $\Lambda_{\mathscr{K}}$ is $\mathscr{E}$. We call it the Eisenstein ideal. It measures the congruences between the Hecke eigenvalues of cusp forms and Klingen-Eisenstein series.

We have:

$$\mathrm{tr}\rho_{\mathbb{T}}(\mathrm{mod} I) \equiv \mathrm{tr}\rho_{\mathbf{E}_{\mathrm{Kling}}}(\mathrm{mod}\mathscr{E}).$$

Let m be the order of $\mathscr{L}_{\pi,\mathscr{K},\xi}\cdot\mathscr{L}_{\xi'}$ at a height one prime P of $\Lambda_{\mathscr{K}}$ which is not a pullback of a height one prime of $\mathscr{O}_L[[\Gamma^+_{\mathscr{K}}]]$. Suppose $m>0$. We now show how this gives information about Eisenstein congruences. Write ℓ for the Fourier-Jacobi functional we constructed. By our assumption on P we have proved that $\ell(\mathbf{H})\not\equiv 0(\mathrm{mod} P)$. Consider the $\Lambda_{\mathscr{K}}$-linear map:

$$\mu:\mathbb{T}\to\Lambda_{\mathscr{K},P}/P^m\Lambda_{\mathscr{K},P}$$

given by: $\mu(t)=\ell(t.\mathbf{H})/\ell(\mathbf{H})$ for t in the Hecke algebra. Then:

$$\ell(t.\mathbf{H})\equiv\ell(t\mathbf{E}_{\mathrm{Kling}})\equiv\lambda(t)\ell(\mathbf{E}_{\mathrm{Kling}})\equiv\lambda(t)\ell(\mathbf{H})(\mathrm{mod} P^m)$$

so I is contained in the kernel of μ. Thus it induces: $\Lambda_{\mathscr{K},P}/\mathscr{E}\Lambda_{\mathscr{K},P}\twoheadrightarrow\Lambda_{\mathscr{K},P}/P^m\Lambda_{\mathscr{K},P}$. In other words the order of the Eisenstein ideal at P is at least m.

Galois Argument

- We now turn to the Galois theoretic "lattice construction". It first appeared in a simple form in a work of Ribet in 1980's, called "Ribet's lemma". It was later on greatly developed by Wiles. However Wiles' approach is extremely complicated. Here we present a later simplified method.
- To illustrate we first give the construction in Wiles' context. Let $\mathbb{I}$ be a reduced ring which is a finite Λ-algebra. Let ρ be a Galois representation of $G_{\mathbb{Q}}$ on $\mathbb{I}^2$. Let J and I be nonzero ideals of Λ and $\mathbb{I}$ respectively, such that the structure map induces $\Lambda/J \simeq \mathbb{I}/I$. Let P be a height one prime of Λ such that $\mathrm{ord}_P(J) = t > 0$. Then there is a unique height one prime P' of $\mathbb{I}$ containing (I, P). Since $\mathbb{I}$ is reduced we can talk about its total fraction ring $K = \prod_i F_{\mathbb{J}_i}$ where $\mathbb{J}_i$'s are domains finite over $\mathbb{I}$ and the $F_{\mathbb{J}_i}$'s are fraction fields of the $\mathbb{J}_i$'s.

We axiomize the situation: suppose

1. Each representation $\rho_{\mathbb{J}_i}$ on $F_{\mathbb{J}_i}^2$ via the projection is irreducible;
2. There are $\Lambda^\times$-valued characters χ_1 and χ_2 of $G_\mathbb{Q}$ such that
$$\mathrm{tr}\rho(\sigma) \equiv \chi_1(\sigma) + \chi_2(\sigma) (\bmod I);$$
3. There are $\mathbb{I}^\times$-valued characters χ_1' and χ_2' of $G_{\mathbb{Q}_p}$ such that
$$\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \chi_1' & * \\ & \chi_2' \end{pmatrix}$$
and there is a $\sigma_0 \in G_{\mathbb{Q}_p}$ such that
$$\chi_1'(\sigma_0) \not\equiv \chi_2'(\sigma_0) (\bmod P').$$

4 For each $\sigma \in \mathbb{I}[G_{\mathbb{Q}_p})]$,

$$\chi_1(\sigma) \equiv \chi_1'(\sigma)(\mod I), \chi_2(\sigma) \equiv \chi_2'(\sigma)(\mod I).$$

5 ρ is unramified outside p.

We define the Selmer module $X := H^1_{\mathrm{ur}}(\mathbb{Q}, \Lambda^*(\chi_1^{-1}\chi_2))^*$. Here ur means unramified everywhere and $*$ means Pontryagin dual. We prove

Proposition

Under the assumptions above,

$$\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) \geq \mathrm{ord}_P(J).$$

We recall the definition of characteristic ideal. Suppose A is a Noetherian normal domain and M a finitely generated A-module. The characteristic ideal of M as an A module $\mathrm{char}_A(M)$ consists of elements $x \in A$ such that for any height one prime P of A,

$$\mathrm{ord}_P(x) \geq \mathrm{length}_{A_P}(M_P).$$

If M is not torsion, then its characteristic ideal is defined to be 0.

Proof

We try to find an $\mathbb{I}$ lattice in $\rho_{\mathbb{I}} \otimes F_{\mathbb{I}}$, whose reduction modulo $\mathscr{L}_p$ becomes a non-trivial extension $\begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$.

We prove the above proposition. Suppose t is the order of J at P (> 0). We take the σ_0 in assumption (3) and a basis (v_1, v_2) so that $\rho(\sigma_0)$ has the form $\mathrm{diag}(\chi_1(\sigma_0), \chi_2(\sigma_0))$. We write

$$\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in M_2(K)$$

for each $\sigma \in K[G_{\mathbb{Q}}]$ with respect to this basis. Then we claim the following: let $r := \chi_1(\sigma_0) - \chi_2(\sigma_0)$, (so $r \notin P$)

a ra_σ, rd_σ, $r^2 b_\sigma c_\tau \in \mathbb{I}$ for all σ, τ in $\mathbb{I}[G_\mathbb{Q}]$, and $ra_\sigma \equiv r\chi_1(\sigma)(\mod I)$, $rd_\sigma \equiv r\chi_2(\sigma)(\mod I)$ and $r^2 b_\sigma c_\tau \equiv 0(\mod I)$;

b $\mathscr{C} := \{c_\sigma, \sigma \in \mathbb{I}[G_\mathbb{Q}]\}$ is a finite faithful Λ-module.

c $c_\sigma = 0$ for all $\sigma \in I_p$.

The c follows from (3) and b follows from (1) and $G_\mathbb{Q}$ being compact. The a is by calculation, e.g. set $\delta_1 = \sigma_0 - \chi_2(\sigma_0)$, then $ra_\sigma = \mathrm{tr}\rho(\delta_1\sigma) = r\chi_1(\sigma)(\mod I)$ by assumption (2).

Now we deduce the proposition from these claims.

- Let $\mathscr{M} := \mathbb{I}_P[G_{\mathbb{Q}}]v_1 \subset V$. Then it is easy to check that $\mathscr{M} = \mathbb{I}_P v_1 + \mathscr{C}_P v_2$. Define $\mathscr{M}_2 = \mathscr{C}_p v_2$.
- Then (a) ($r^2 b_\sigma c_\tau \equiv 0 (\mod I)$) implies that $\bar{\mathscr{M}_2} := \mathscr{M}_2 / I\mathscr{M}_2 \subset \bar{\mathscr{M}} := \mathscr{M}/I\mathscr{M}$ is a direct summand which is stable under $G_{\mathbb{Q}}$.
- Define $\mathscr{M}_1 = \mathbb{I}_P v_1$ and $\bar{\mathscr{M}_1} = \mathscr{M}_1 / I\mathscr{M}_1$, then

$$0 \to \bar{\mathscr{M}_2} \to \bar{\mathscr{M}} \to \bar{\mathscr{M}_1} \to 0.$$

Now we return to the lattice construction before localization at P. We will find a finite Λ-module $\mathfrak{m}_2 \subset \bar{\mathscr{M}}_2$ such that

i $\mathfrak{m}_{2,P} = \bar{\mathscr{M}}_2$;

ii There exists a Λ-map $X \to \mathfrak{m}_2$ that is a surjection after localization at P.

We let $\mathfrak{m} \subset \bar{\mathscr{M}}$ be the submodule generated by $\mathbb{I}[G_{\mathbb{Q}}]v_1$. Let $\mathfrak{m}_2 = \mathfrak{m} \cap \bar{\mathscr{M}}_2$ and $\mathfrak{m}_1 := \mathfrak{m}/\mathfrak{m}_2 \subset \bar{\mathscr{M}}_1$. Then from the construction we have $\mathfrak{m}_{2,P} = \bar{\mathscr{M}}_2$.

We have

$$(*)\ 0 \to \mathfrak{m}_2 \to \mathfrak{m} \to \mathfrak{m}_1 \to 0.$$

- By (5) and (c) the above extension is everywhere unramified, i.e. is the split extension when restricting to I_v for every prime v.
- $\bar{\mathscr{M}}_1$ is isomorphic to $\Lambda_P/P^t\Lambda_P$, it is easy to show that $\mathfrak{m}_1$ is isomorphic to $\Lambda/P^t\Lambda$.
- By (a) the actions of $G_{\mathbb{Q}}$ on $\mathfrak{m}_2$ and $\mathfrak{m}_1$ are given by χ_2 and χ_1 respectively.

We expect the exact sequence (*) gives the required extension. More precisely let $[\mathfrak{m}] \in H^1(G_{\mathbb{Q}}, \mathfrak{m}_2(\chi_2\chi_1^{-1}))$ be the extension class defined by (*). Then we get a canonical map

$$\theta : \mathrm{Hom}_\Lambda(\mathfrak{m}_2, \Lambda^*) \to H^1(G_{\mathbb{Q}}, \Lambda^*(\chi_2\chi_1^{-1})).$$

Take the Pontryagin dual, we get

$$\theta^* : H^1(G_{\mathbb{Q}}, \Lambda^*(\chi_2\chi_1^{-1}))^* \to \mathrm{Hom}_\Lambda(\mathfrak{m}_2, \Lambda^*)^* \simeq \mathfrak{m}_2.$$

(the last isomorphism is by a straightforward check).

We claim that the θ^* becomes a surjection after localization at P. To see the claim, by Pontryagin duality it is enough to show that $\mathscr{R} := \ker(\theta) = 0$ after localization at P. For any finite subset S of $\mathscr{R}$, let

$$\mathfrak{m}_S := \cap_{\phi \in S} \ker(\phi).$$

Then

$$0 \to \mathfrak{m}_2/\mathfrak{m}_S \to \prod_{\phi \in S} \Lambda^* \to (\prod_{\phi \in S} \Lambda^*)/(\mathfrak{m}_2/\mathfrak{m}_S) \to 0$$

where the first map is given by the product of all $\phi \in S$.

We equip each module in the above exact sequence with the action of $G_{\mathbb{Q}}$ given by $\chi_2\chi_1^{-1}$, and take the cohomology long exact sequence. We see that the kernel of

$$H^1(G_{\mathbb{Q}}, \mathfrak{m}_2/\mathfrak{m}_S(\chi_2\chi_1^{-1})) \to H^1(G_{\mathbb{Q}}, \prod_{\phi\in S}\Lambda^*(\chi_2\chi_1^{-1}))$$

is killed by $1-\chi_2\chi_1^{-1}(\sigma_0)$ which is not in P. On the other hand by definition this kernel contains the image of the class $[\mathfrak{m}]$. So the extension

$$0 \to \mathfrak{m}_{2,P}/\mathfrak{m}_{S,P} \to \mathfrak{m}_P/\mathfrak{m}_{S,P} \to \mathfrak{m}_{1,P} \to 0$$

is split.

If $(\frac{\mathfrak{m}_2}{\mathfrak{m}_S})_P$ were nonzero, this contradicts the fact that $\mathfrak{m}$ is generated by $\bar{v}_1$ over $\mathbb{I}[G_{\mathbb{Q}}]$. So $\mathfrak{m}_{2,P} = \mathfrak{m}_{S,P}$, thus $\mathscr{R}_P = 0$ by arbitrariness of S. This proves the claim, and thus also (ii) above. We can now prove the proposition.

Definition

Let M be a finitely presented torsion module over the ring A. Take a finite presentation as

$$A^r \to A^s \to M \to 0$$

for $r \geq s$, we define the Fitting ideal $\mathrm{Fitt}_A(X)$ to be the ideal of A generated by the $s \times s$ minors of the matrix for the above presentation. By linear algebra it does not depend on the choice of the finite presentation (we omit the proof). Clearly Fitting ideal respects arbitrary base change.

End of Proof

We have from the claim the inequality

$$\begin{aligned}\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) &= \mathrm{ord}_P(\mathrm{char}_{\Lambda_P}(X_P)) = \mathrm{ord}_P(\mathrm{Fitt}(\Lambda_P/\mathrm{char}_\Lambda(X_P))) \\ &\geq \mathrm{ord}_P(\mathrm{Fitt}_{\Lambda_P}(\mathfrak{m}_{2,P})) = \mathrm{ord}_P(\mathrm{Fitt}_{\Lambda_P}(\bar{\mathscr{M}}_2)).\end{aligned}$$

But

$$\begin{aligned}\mathrm{Fitt}_{\Lambda_P}\bar{\mathscr{M}}_2(\bmod J) &= \mathrm{Fitt}_{\Lambda_P/J\Lambda_P}\bar{\mathscr{M}}_2 = \mathrm{Fitt}_{\mathbb{I}_P/I\mathbb{I}_P}\bar{\mathscr{M}}_2 \\ &= \mathrm{Fitt}_{\mathbb{I}}\bar{\mathscr{M}}_2(\bmod I) = \mathrm{Fitt}_{\mathbb{I}}\mathscr{M}_2(\bmod I) = 0,\end{aligned}$$

where the last equality follows from the fact that $\mathscr{M}_2$ is a faithful module over $\mathbb{I}$, and Fitting ideal annihilates the module. So $\mathrm{Fitt}_{\Lambda_P}(\bar{\mathscr{M}}_2) \subseteq J$, and $\mathrm{ord}_P(\mathrm{char}_\Lambda(X)) \geq \mathrm{ord}_P J$. We proved the proposition.

Now return to our situation.

- We have 3 irreducible components for the Galois representation $\begin{pmatrix} \chi & & \\ & \rho & \\ & & \chi' \end{pmatrix}$ associated to Klingen parabolic subgroup, (the χ and χ' are characters and ρ has rank two).
- congruence to cusp forms modulo the *p*-adic *L*-function produces extensions in $\begin{pmatrix} \chi & *1 & *2 \\ & \rho & \\ & & \chi' \end{pmatrix}$, where the $*1$ is the extension class we are looking for, and $*2$ is an extension of χ by χ'. In the lattice construction there is some complicated interaction between $*1$ and $*2$, making the proof more difficult than the IMC for number fields.

- Write R for the reduced semi-ordinary cuspidal Hecke algebra and I for the Eisenstein congruence ideal. Under certain residual distinguished condition, we can pick up a group algebra element as before and similarly construct a lattice L as generated by a χ-eigenvector for this element over $R[G]$ and use certain idempotents to obtain a direct summand decomposition of R-modules.

$$L = L(\chi') \oplus L(\rho) \oplus L(\chi) = L(\alpha_1) \oplus L(\alpha_2) \oplus L(\alpha_3) \oplus L(\alpha_4)$$

where α_i are eigenvalues for this group algebra element, and α_2 and α_3 correspond to the ρ-part.

Under this decomposition the group algebra $R[G]$ can be written as matrix form

$$\begin{pmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \mathscr{A}_{13} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \mathscr{A}_{23} \\ \mathscr{A}_{31} & \mathscr{A}_{32} & \mathscr{A}_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & M_{1\times 2}(A_{12}) & A_{13} \\ M_{2\times 1}(A_{21}) & M_2(A_{22}) & M_{2\times 1}(A_{23}) \\ A_{31} & M_{1\times 2}(A_{32}) & A_{33} \end{pmatrix}$$

where the A_{ij}'s are modules over R (fractional ideals of its total fraction ring), and each entry can be regarded as homomorphisms between the $L(?)$'s. This is called generalized matrix algebra (GMA) (a good reference is Chapter 1 in the book of Bellaiche-Chenevier).

For simplicity we only consider the localized picture, i.e. the construction over R_P where P is a height one prime. Moreover we have the following.

Lemma

$$\mathscr{A}_{\mu,\lambda}(x)\mathscr{A}_{\lambda,\mu}(y) \in \mathrm{End}(L(\mu), L(\mu))$$

has image contained in $IL(\mu)$ *for each* λ, μ *in* $\{\chi', \rho, \chi\}$ *and each* x, y *in* $R[G]$*. Also we have the image of* $\mathscr{A}_{\chi,\mu}(x)$ *is contained in* $IL(\chi)$ *whenever* $\mu \neq \chi$*.*

This the analogue of $r^2 b_\sigma c\tau \equiv 0 (\mathrm{mod}\ I)$ in the previous example.

Let $\mathscr{L} := L/IL$ and $\mathscr{L}(\mu) := L(\mu)/IL(\mu)$ ($\mu = \chi'$, ρ or χ). Recall $\mathscr{A}_{\mu,\lambda}$ is regarded as an element of $\mathrm{End}_{R/I}(\mathscr{L}(\lambda), \mathscr{L}(\mu))$. We define

$$\mathscr{J}(\chi') = \sum_{x \in R[G]} \mathrm{Im}(\mathscr{A}_{\chi',\rho}(x)) \subset \mathscr{L}(\chi'),$$

$$\mathscr{J}(\rho) = \sum_{x \in R[G]} \mathrm{Im}(\mathscr{A}_{\rho,\chi'}(x)) \subset \mathscr{L}(\rho).$$

The

$$x \mapsto \mathscr{A}_{\rho,\rho}(x) \in \mathrm{End}_R(\mathscr{L}(\rho))$$

gives a G-representation. Moreover, there is a finitely generated R-module $\mathscr{N}$ such that there is an isomorphism

$$(\mathscr{L}(\rho), \mathscr{A}_{\rho,\rho}) = (\mathscr{N} \otimes_R N, 1 \otimes \rho).$$

There is a R-submodule $\mathscr{N}'$ of $\mathscr{N}$ such that the above isomorphism induces an isomorphism

$$(\mathscr{J}(\rho), \mathscr{A}_{\rho,\rho}) = (\mathscr{N}' \otimes N, \mathscr{A}_{\rho,\rho}).$$

Thus there is an isomorphism of $R[G]$-modules

$$\mathscr{L}(\rho)/\mathscr{J}(\rho) = (\mathscr{N}/\mathscr{N}') \otimes N.$$

One shows as before that the $\mathscr{L}(\chi')\oplus\mathscr{J}(\rho)$, $\mathscr{J}(\chi')\oplus\mathscr{L}(\rho)$ and $\mathscr{L}(\chi')\oplus\mathscr{L}(\rho)$ are both $R[G]$-modules. We also have the exact sequence

$$0\to\frac{\mathscr{L}(\rho)\oplus\mathscr{L}(\chi')}{\mathscr{J}(\rho)\oplus\mathscr{L}(\chi')}\to\frac{\mathscr{L}(\chi')\oplus\mathscr{L}(\rho)\oplus\mathscr{L}(\chi)}{\mathscr{J}(\rho)\oplus\mathscr{L}(\chi')}\to\mathscr{L}(\chi)\to 0. \tag{3.1}$$

One shows that $\mathscr{L}(\chi)$ is isomorphic to R/I, with the induced Galois action given by χ. The induced Galois action on $\frac{\mathscr{L}(\chi')\oplus\mathscr{L}(\rho)}{\mathscr{J}(\rho)\oplus\mathscr{L}(\chi')}$ is given by ρ.
We first consider the case when $L\otimes F$ is rank four over F where F is the total fraction field of R (this means for all irreducible components of $\mathrm{Spec}R$, the corresponding Galois representation is irreducible). In this case from the faithfulness of $L(\alpha_2)$, we see that the Fitting ideal of $\mathscr{N}$ is contained in I.

- Suppose P is a height one prime which is not a pullback of a height one prime of $\mathcal{O}_L[[\Gamma^+]]$. As before we see we may construct a surjection from the the dual Selmer group of $\rho\otimes\chi^{-1}$ to $\mathcal{N}/\mathcal{N}'$ (the local Selmer condition at p follows from our description for Galois representation for semi-ordinary forms). By a similar argument as above, the dual Selmer group of $\chi'\chi^{-1}$ surjects onto $\mathcal{L}(\chi')/\mathcal{J}(\chi')$. But the dual Selmer group of the Dirichlet character χ'/χ has characteristic ideal generated by the Dirichlet p-adic L-function by work of Mazur-Wiles proving the Iwasawa main conjecture for Dirichlet characters. This is a nonzero element in $\mathcal{O}_L[[\Gamma^+_{\mathcal{K}}]]$ (thus co-prime to P). On the other hand from the lemma above, $\mathcal{L}(\chi')/\mathcal{J}(\chi')$ surjects onto $\mathcal{J}(\rho)$. So the latter, and also $\mathcal{N}'$ is zero after localization at P. So the Fitting ideal of the dual Selmer group for $\rho\otimes\chi^{-1}$ is contained in I.

- In practice, fixing the height one prime P, we can delete the irreducible components of $\mathrm{Spec}(R)$ which do not contain the point P. The argument above still works if the associated Galois representation is still irreducible for the remaining components.
- Now suppose we do not have that $L \otimes F$ is rank four over F after deleting the irreducible components of $\mathrm{Spec}(R)$ which do not contain the point P, that is, for some remaining irreducible component of $\mathrm{Spec}(R)$ the associated Galois representation has two irreducible components. In this case, the lattice construction for this irreducible component gives non-trivial P-part of the dual Selmer group for the character $\chi'\chi^{-1}$ (called endoscopic congruences), contradicting again the results of Mazur-Wiles. So we conclude the proof of the main result.

If P does come from a height one prime in the cyclotomic direction, the lattice construction above breaks down and we do not get control on the size of the Selmer module. In this case, one can nevertheless show that if P is a height one prime of the p-adic L-function we study, then the order of the characteristic ideal of the dual Selmer module at P is at least one.

Thanks

Thank You !