An overview to Faltings' proof and LV's proof of Mordell conjecture

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<u>Conventions</u> · Curve: a sm proj geom-conn variety / field of dim 1.

<u>Mordell conj</u>: K number field, C/K curve of genus >1.

Then $\# C(K) < \infty$.

Previously we obtain 3 proofs:

① Faltings 1983 (with Faltings height of AVs)

relative to Shafarevich conj(s) Tate conj(s),

used basic results in p-adic Hodge theory.

essentially without Arakelov geom tools.

② Vojta 1991 (with Diophantine approximation & Arakelov geom).

based on Mordell-Weil thm.

③ Lawrence-Venkatesh 2018

the major tool: p-adic Hodge theory

$\rightsquigarrow$ comparison of p-adic Tate module & Hodge cohom.

without counting argument, with more geometry.

§1 Faltings' proof

<u>Thm (A)</u> (Mordell conj) $C(K)$ is finite.

$\Uparrow$

<u>Thm (B)</u> (Shafarevich conj I) Fix K, $S \subset M_K$ finite, $g \geq 1$.

$\#\{Y/K \text{ curve of genus } g, \text{ with good reductions outside } S\} < \infty$

$\Uparrow$

<u>Thm (C)</u> (Shafarevich conj. II) Fix K, $S \subset M_K$ finite, $g \geq 1$.

$\#\{A/K \text{ AV of dim } g, \text{ with good reductions outside } S\} < \infty$.

Proof idea for (B) ⇒ (A)

By Kodaira-Parshin construction

(to translate information on C to a moduli space $\mathcal{M}_g$).

for C/K curve with $g(C)>1$,

$\exists\ \phi: C' \to C$ finite étale (i.e. everywhere unramified) morphism

(C' another curve with genus $g(C')>1$).

with $\pi: X \to C'$ proj. smooth morphism whose fibers are curves of genus >1.

$$\begin{array}{ccc} X_s & \hookrightarrow & X \\ \downarrow & & \downarrow \pi \\ s \in & C' & \xrightarrow{\phi} C. \end{array}$$

<u>Then</u> $C(K)$ finite $\overset{?}{\Leftarrow}$ $C'(K')$ finite (K'/K <u>some</u> finite ext'n).

$$\Leftarrow \#\left\{\begin{array}{l}\text{Curves } X_s/K' \text{ of genus } g(X_s)>1, \\ \text{with good reductions outside } S \subset M_K \text{ (finite set)}\end{array}\right\} < \infty$$

<u>Lemma</u> $\phi: C' \to C$ finite étale $/K$, with $g', g > 1$.

Then $\forall K'/K$ finite, $C(K')$ finite

$\Leftrightarrow \forall K''/K$ finite, $C'(K'')$ finite.

← C' is pretended to be the moduli space $\mathcal{M}_{g';K}$.

($\exists$ finite morphism $C' \to \mathcal{M}_{g';K}$ i.e. any fiber$/\bar{K}$ is isom. to at most finitely many fibers.)

<u>Proof</u> "⇒" trivial.

"⇐" need to prove: $\exists$ fin ext'n K'/K s.t.

$C'(K')$ finite ⇒ $C(K)$ finite

$$\begin{array}{ccc} C' & \hookrightarrow & \mathcal{C}' \\ \phi \downarrow & & \downarrow \psi \\ C & \hookrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & U \underset{\text{open}}{\subseteq} \operatorname{Spec} \mathcal{O}_K \end{array}$$

integral models

$\mathcal{C}'/U$, $\mathcal{C}/U$ integral models of C', C.

Can require $\mathcal{C}'/U$, $\mathcal{C}/U$ smooth proj

$\psi: \mathcal{C}' \to \mathcal{C}$ smooth proj.

(by shrinking U if necessary)

- $s \in C(K)$ rational pt (closed pt)

$\rightsquigarrow \phi^{-1}(s) \subset C'$ finite set of closed pts in C'.

$$\begin{tikzcd} \psi^{-1}(\tilde{S}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d, "\psi"] \\ \tilde{S} \arrow[r, hook] & \mathcal{C} \end{tikzcd}$$

$\tilde{S} \subset \mathcal{C}$ Zariski closure of S

ψ finite étale

$\Rightarrow \psi^{-1}(\tilde{S}) \to \tilde{S}$ finite étale, $\tilde{S} \cong U \subset \operatorname{Spec} \mathcal{O}_K$.

(extends $\psi^{-1}(s) \longrightarrow s$), $\psi^{-1}(s) = \coprod_i \operatorname{Spec} K_i$, $s = \operatorname{Spec} K$

$\Rightarrow$ $\psi^{-1}(s)$ is unramified over K above all $v \in U$.

Upshot: every pt of $\phi^{-1}(s)(\bar{K}) \subset C'(\bar{K})$ is defined over a finite ext'n M/K of bounded deg, unram above U.

· By Hermite's thm, there are only finitely many such M.

Then can find fin ext'n K'/K containing all such M.

$\Rightarrow$ $C(K) \subset \phi(C'(K'))$ finite.

Proof of (C) $\Rightarrow$ (B)

$\{Y/K$ curve of genus $g>1$, with good red outside $S\}$

$\big\downarrow$ (Jacobian) (inj. by Torelli's thm)

$\{A/K$ prin polar AV of dim g, with good red outside $S\}$

$\big\downarrow$ (finite-to-one)

$\{A/K$ AV of dim g, with good red outside $S\}$.

Sketch proof of (C) (Counting AVs)

Tool: Faltings' height.

Def'n (1) A/K AV / number field. The Faltings height

$$h(A) := \frac{1}{[K:\mathbb{Q}]} \widehat{\deg}(\underline{\omega}_A) \quad \text{(arith deg)}$$

where $\pi : \mathcal{A} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ Néron model

$e : \operatorname{Spec} \mathcal{O}_K \to \mathcal{A}$ id section,

$\underline{\omega}_{\mathcal{A}} := e^* \omega_{\mathcal{A}/\mathcal{O}_K} = e^*(\det \Omega_{\mathcal{A}/\mathcal{O}_K})$ line bundle on $\operatorname{Spec} \mathcal{O}_K$.

(2) Faltings metric: $\forall \sigma : K \hookrightarrow \mathbb{C}$,

$\alpha \in \underline{\omega}_{\mathcal{A}} \otimes_\sigma \mathbb{C} \cong \Gamma(A_\sigma(\mathbb{C}), \omega_{A_\sigma(\mathbb{C})/\mathbb{C}})$.

$\|\alpha\|^2_{\mathrm{Fal}} = \left| \int_{A_\sigma(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|$.

Thm(H) (Northcott) Fix K, g, H.

$\#\{A/K \mid$ semistable reduction, $\dim g$, $h(A) < H\} < \infty$

$\rightsquigarrow$ Then we want: $\boxed{h(A) < \text{const}}$

Thm(D) (Finiteness I) Fix A/K AV.

$\#\{B \text{ AV}/K \mid B \overset{\text{isog}}{\sim} A\}/(\text{isom}) < \infty$.

Thm(E) (Finiteness II) Fix K, g. $S \subset M_K$ finite.

$\#\{A/K$ of $\dim g$ with good red outside $S\}/(\text{isogeny}) < \infty$.

Then (D)+(E) $\Rightarrow$ (C).

Faltings introduced (D') (a weaker ver of (D))

to get $\quad$ Thm (H) $\Rightarrow$ Thm (D') $\Rightarrow$ $\left.\begin{array}{l}\text{Thm (F)} + \text{(G)} \\ \text{Thm (H)}\end{array}\right\} \Rightarrow$ Thm (D)

& $\quad \boxed{\text{Thm (F)} + \text{(G)} \Rightarrow \text{Thm (E)}}$. $\leftarrow$ important for us.

Thm(F) (Semisimplicity) A/K AV/number field.

The G_K-rep'n $V_\ell(A)$ is semisimple.

Recall Tate module $T_\ell(A) := \varprojlim_n A(\bar{K})[\ell^n]$

& $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Thm (G) (Tate conj) $A, B/K$ AVs over a number field.

$$\mathrm{Hom}_K(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\bar{K}/K)}(V_\ell(A), V_\ell(B)).$$

Proof of (F) + (G) $\Rightarrow$ (E)

By Thm (G), counting AV becomes counting Galois rep'ns $G_K \curvearrowright V_\ell(A)$.

Thm (Faltings) Fix K, g, ℓ, $S \subset M_K$ finite.

Require that S contains $\{v \mid \ell\}$.

Then $\exists$ only finitely many semisimple G_K-rep'n (V, ρ), where

① V v.s. $/\mathbb{Q}_\ell$ of dim $2g$.

② V pure of wt 1 outside S

i.e. $\forall v \notin S$ non-arch, (V, ρ) is unram at v,

and the char poly of the Frobenius Frob_v

has integer coefficients $\in \mathbb{Z}$, all roots have absolute value $q_v^{\frac{1}{2}}$ in $\mathbb{C}$.

$$\begin{array}{ccccc} I_v & \hookrightarrow & G_v & \hookrightarrow & G_K \\ \uparrow & & \| & & \\ \text{inertia grp} & & G_{K_v} & \leftarrow \text{decomp grp} & \end{array}$$

unram: I_v acts trivially as id on V — residue field at v

Frob_v = lifting of Frob of $\mathrm{Gal}(\overline{k(v)}/k(v))$ in G_v.

Pf Step1 Convert (V, ρ) to trace $\mathrm{tr}(\rho) : G_K \to \mathbb{Q}_\ell$

Step2 For fixed v, $\mathrm{tr}(\rho(\mathrm{Frob}_v))$ has only finitely many choices (by ②)

Step3 Chebotarev density thm:

only need to control a finite set of Frob_v's

§2 LV's proof

C/K with $g_0 > 1$. Want $C(K)$ to be finite.

- By Kodaira-Parshin construction,

  up to replacing C by a finite étale cover,

  can assume $\exists\, \pi : X \to C$ family of curves of genus $g > 1$.

  $\exists\, S \subset M_K$ finite s.t. $\forall\, s \in C(\bar{K})$, X_s/K has good reduction outside S.

- Consider

$$\begin{tikzcd} C(K) \arrow[r, "\Phi"] \arrow[d, "i_v\ (\mathrm{inj})"'] & \{G_K\text{-rep'n} / \mathbb{Q}_p \text{ of dim } 2g,\ \text{pure of wt } 1 \text{ outside } S\} \arrow[d, "j_v"] \\ C(K_v) \arrow[r, "\Phi_v"] & \{G_{K_v}\text{-rep'n} / \mathbb{Q}_p \text{ of dim } 2g\} \end{tikzcd}$$

$$s \longmapsto H^1_{et}(X_{s,\bar{K}}, \mathbb{Q}_p)$$

Choose v place of K above p, $v \in S$ (can enlarge S if necessary)

$\rightsquigarrow$ X_s/K good red at v and v unram above p.

Key thm (LV) Φ_v is finite-to-one

$\rightsquigarrow$ Assuming key thm, the Mordell is as follows:

Φ_v is finite-to-one

$\Rightarrow$ $\Phi_v \circ i_v = j_v \circ \Phi$ is finite-to-one

$\Rightarrow$ Φ is finite-to-one.

By Faltings' thm, $\#\{G_K\text{-rep'n}\}$ finite.

Then $C(K)$ is finite.

Rmk (1) Faltings' thm says the G_K-rep'n

$$V_p(A) = H^1_{et}(A_{\bar{K}}, \mathbb{Q}_p)^\vee \text{ is semisimple.}$$

But LV doesn't assume it.

In fact, LV proved the semisimplicity for all but finitely many $s \in C(K)$.

(2) Φ_v is purely local.

Recall p-adic Hodge theory:

(1) (Comparison) Y/K_v curve ($Y = X_s$)

$$H^1_{et}(\bar{Y}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^1_{dR}(Y/K_v) \otimes_{K_v} B_{dR}$$

$$H^1_{et}(\bar{Y}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^1_{cris}(Y_{k_v}/K_v) \otimes_{K_v} B_{cris}$$

(2) (Equivalence of categories)

H^1_{et}-side $\{G_{K_v}\text{-rep } W : \text{with } \mathbb{Q}_p\text{-coeffs}\}$ $\qquad W$

$\downarrow \qquad\qquad \mapsto$

H^1_{dR}-side $\left\{ \begin{array}{l} \text{filtered } \phi\text{-module} \\ (V, \phi, \mathrm{Fil}^1) \end{array} \middle| \begin{array}{l} V: 2g\text{-dim'l } K_v\text{-space} \\ \phi: V \to V \text{ semilinear map} \\ \mathrm{Fil}^1 V \subset V \ g\text{-dim'l subspace} \end{array} \right\}$ $\qquad (W \otimes_{\mathbb{Q}_p} B_{dR})^{G_{K_v}}$

$\wr\wr$

H^1_{cris}

Then
$$\begin{tikzcd} C(K_v) \arrow[r, "\Phi_v"] \arrow[rr, bend right, "\Phi_v'"'] & \{G_{K_v}\text{-rep'ns}\} \arrow[r, "\sim"] & \{\text{filtered } \phi\text{-modules}\}/(\text{isom}) \end{tikzcd}$$

take a residue disc $D \subset C(K_v)$

(all $s \in D$ give fiber X_s with the same reduction).

$\Rightarrow$ it trivializes crystalline cohomology with ϕ-action.

Can assume $(V, \phi) = (V_0, \phi_0)$.

$$\begin{tikzcd} D \arrow[r, "\Phi_v'"] \arrow[dr, "\Phi_v''"'] & \{(V_0, \phi_0, \mathrm{Fil}^1)\}/\left(\substack{\text{automorphism} \\ (V_0,\phi_0) \xrightarrow{\sim} (V_0,\phi_0)}\right) \\ & \{(V_0, \mathrm{Fil}^1)\} \arrow[u, "\eta"'] \end{tikzcd}$$

$2g \quad g$-dim'l

Need: Φ_v finite-to-one.

Two steps (1) $\Phi_v'' : D \longrightarrow Gr(K_v)$ ("analytic") is finite-to-one.

p-adic manifold $\quad$ algebraic variety

(Gr = Grassmannian $/K_v$).

(2) $(\mathrm{im}\, \Phi_v'') \cap (\text{fiber of } \eta)$ finite, for any fiber.

fibers of q are orbits of centralizer $Z(\phi)$ of ϕ in $GL(V_0)$.
Need $Z(\phi)$ "small".