

An overview to Faltings' proof and Liu's proof of Mordell conjecture

Xinyi Yuan

Conventions · Curve: a sm proj geom-conn variety / field of dim 1.

Mordell conj. K number field, C/K curve of genus > 1 .

Then $\#C(K) < \infty$.

Previously we obtain 3 proofs:

- ① Faltings 1983 (with Faltings height of AVs)
relative to Shafarevich conj(s), Tate conj(s),
used basic results in p -adic Hodge theory.
essentially without Arakelov geom tools.
- ② Vojta 1971 (with Diophantine approximation & Arakelov geom).
based on Mordell-Weil thm.
- ③ Lawrence-Venkatesh 2018
the major tool: p -adic Hodge theory
 \rightarrow comparison of p -adic Tate module & Hodge coh. \mathbb{Q}
without counting argument, with more geometry.

§1 Faltings' proof

Thm (A) (Mordell conj) $C(K)$ is finite.

\Uparrow
Thm (B) (Shafarevich conj I) Fix $K, S \subset M_K$ finite, $g \geq 1$.

\Uparrow $\# \{Y/K \text{ curve of genus } g, \text{ with good reductions outside } S\} < \infty$

Thm (C) (Shafarevich conj. II) Fix $K, S \subset M_K$ finite, $g \geq 1$.

$\# \{A/K \text{ AV of dim } g, \text{ with good reductions outside } S\} < \infty$.

Proof idea for (B) \Rightarrow (A)

(to translate information on C to a moduli space \mathcal{M}_g).

By Kodaira-Parshin construction

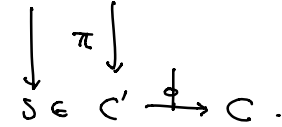
for C/K curve with $g(C) > 1$,

$\exists \phi: C' \rightarrow C$ finite étale (i.e. everywhere unramified) morphism

(C' another curve with genus $g(C') > 1$). $X_S \leftrightarrow X$

with $\pi: X \rightarrow C'$ proj. smooth morphism

whose fibers are curves of genus > 1 .



Then $C(K)$ finite $\Leftrightarrow C'(K')$ finite (K'/K some finite ext'n).

$$\Leftrightarrow \# \left\{ \begin{array}{l} \text{Curves } X_S/K \text{ of genus } g(X_S) > 1, \\ \text{with good reductions outside } S \subset M_K \text{ (finite set)} \end{array} \right\} < \infty$$

Lemma $\phi: C' \rightarrow C$ finite étale $/K$, with $g', g > 1$.

Then $\forall K'/K$ finite, $C(K')$ finite

$\Leftrightarrow \forall K''/K$ finite, $C(K'')$ finite.

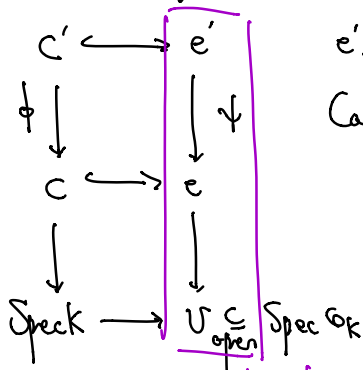
$\leftarrow C'$ is pretended to be the moduli space $\mathcal{M}_{g',K}$.

(\exists finite morphism $C' \rightarrow \mathcal{M}_{g',K}$ i.e. any fiber $/K$ is isom. to at most finitely many fibers.)

Proof " \Rightarrow " trivial.

" \Leftarrow " need to prove: \exists fin ext'n K'/K s.t.

$C'(K') \text{ finite} \Rightarrow C(K) \text{ finite}$



$e'/U, e/V$ integral models of C', C .

Can require $e'/U, e/V$ smooth proj

$\psi: e' \rightarrow e$ smooth proj.

(by shrinking U if necessary)

integral models

$\bullet s \in C(K)$ rational pt (closed pt)

$\hookrightarrow \phi^{-1}(s) \subset C'$ finite set of closed pts in C' .

$$\begin{array}{ccc}
 \psi^+(S) \hookrightarrow e' & \tilde{S} \subseteq e \text{ Zariski closure of } S & \\
 \downarrow \square \downarrow & \psi \text{ finite étale} & \\
 \tilde{S} \hookrightarrow e & \Rightarrow \psi^+(\tilde{S}) \rightarrow \tilde{S} \text{ finite étale} & \\
 & \text{"} \text{ } \text{"} & \\
 & U \subseteq \text{Spec } \mathbb{C}_K &
 \end{array}$$

$$\begin{array}{ccc}
 \text{(extends } \psi^+(s) \rightarrow s \text{)} & & \\
 \text{"} \text{ } \text{"} & & \\
 \text{Spec } K & & \text{Spec } \mathbb{C}
 \end{array}$$

$\Rightarrow \psi^+(s)$ is unramified over K above all $v \in U$.

Upshot every pt of $\psi^+(s)(\bar{K}) \subset C(\bar{K})$ is defined over a finite ext'n M/K of bounded deg, unram above U .

• By Hermite's thm, there are only finitely many such M .

Then can find fin ext'n K'/K containing all such M .

$\Rightarrow C(K) \subset \phi(C(K'))$ finite.

Proof of (A) \Rightarrow (B)

{ γ/K curve of genus $g > 1$, with good red outside S }

(inj. by Torelli's thm) \downarrow (Jacobian)

{ A/K prin polar AV of dim g , with good red outside S }

\downarrow (finite-to-one)

{ A/K AV of dim g , with good red outside S }.

Sketch proof of (C) (Counting AVs)

Tool: Faltings' height.

Def: (i) A/K AV / number field. The Faltings height

$$h(A) := \frac{1}{[K:\mathbb{Q}]} \deg(\omega_A) \text{ (arith deg)}$$

where $\pi: \mathcal{A} \rightarrow \text{Spec } \mathcal{O}_K$ Néron model

$e: \text{Spec } \mathcal{O}_K \rightarrow \mathcal{A}$ id section,

$\omega_{\mathcal{A}} := e^* \omega_{\mathcal{A}/\mathcal{O}_K} = e^*(\det \Omega_{\mathcal{A}/\mathcal{O}_K})$ line bundle on $\text{Spec } \mathcal{O}_K$.

(a) Faltings metric: $\forall \sigma: k \rightarrow \mathbb{C}$,

$$\alpha \in \omega_{\mathcal{A}} \otimes_{\sigma} \mathbb{C} \cong \Gamma(A_{\sigma}(\mathbb{C}), \omega_{A_{\sigma}(\mathbb{C})/\mathbb{C}}).$$

$$\|\alpha\|_{\text{Fal}}^2 = \left| \int_{A_{\sigma}(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|.$$

Thm (H) (Northcott) Fix K, g, H .

$\{A/K \mid \text{semistable reduction, } \dim g, h(A) < H\} < \infty$

\Rightarrow Then we want: $h(A) < \text{const}$

Thm (D) (Finiteness I) Fix A/K AV.

$\{B \text{ AV}/K \mid B \stackrel{\text{isog}}{\sim} A\} / (\text{isom}) < \infty$.

Thm (E) (Finiteness II) Fix $K, g, S \subset M_K$ finite.

$\{A/K \text{ of } \dim g \text{ with good red outside } S\} / (\text{isogeny}) < \infty$.

Then (D)+(E) \Rightarrow (c).

Faltings introduced (D') (a weaker ver of (D))

to get $\left. \begin{array}{l} \text{Thm (H)} \Rightarrow \text{Thm (D')} \Rightarrow \text{Thm (F)+(G)} \\ \text{Thm (H)} \end{array} \right\} \Rightarrow \text{Thm (D)}$

$\&$ $\boxed{\text{Thm (F)+(G)} \Rightarrow \text{Thm (E)}}$. \leftarrow important for us.

Thm (F) (Semisimplicity) A/K AV/number field.

The G_K -rep'n $V_{\lambda}(A)$ is semisimple.

Recall Tate module $T_{\ell}(A) := \varprojlim_n A(\bar{K})[\ell^n]$

$$\& V_{\ell}(A) = T_{\ell}(A) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}.$$

Thm (G) (Tate conj) $A, B/K$ AVs over a number field.

$$\text{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell} \xrightarrow{\sim} \text{Hom}_{\text{Gal}(\bar{K}/K)}(V_{\ell}(A), V_{\ell}(B)).$$

Proof of (F) + (G) \Rightarrow (E)

By Thm (G), counting AV becomes counting Galois reps $G_K \hookrightarrow V_{\ell}(A)$.

Thm (Faltings) Fix $K, g, \ell, S \subset M_K$ finite.

Require that S contains $\{v, \ell\}$.

Then \exists only finitely many semisimple G_K -rep'n (V, ρ) , where

① V v.s. \mathbb{Q}_{ℓ} of dim $2g$.

② V pure of wt 1 outside S

i.e. $\forall v \notin S$ non-arch, (V, ρ) is unram at v ,

and the char poly of the Frobenius Frobr

has integer coefficients $\in \mathbb{Z}$, all roots have absolute value $q_v^{\frac{1}{2}}$ in \mathbb{C} .

$$\begin{array}{ccccc} I_v & \hookrightarrow & G_v & \hookrightarrow & G_K \\ \uparrow & & \uparrow & & \\ \text{inertia grp} & & G_{K_v} & \leftarrow & \text{decomp grp} \end{array}$$

unram: I_v acts trivially as id on V residue field at v

Frobr = lifting of Frobr of $\text{Gal}(\bar{K}(v)/K(v))$ in G_v .

Pf Step 1 Convert (V, ρ) to trace $\text{tr}(\rho): G_K \rightarrow \mathbb{Q}_{\ell}$

Step 2 For fixed v , $\text{tr}(\rho(\text{Frobr}_v))$ has only finitely many choices (by ②)

Step 3 Chebotarev density thm:

only need to control a finite set of Frobr's

§2 LV's proof

C/K with $g_0 > 1$. Want $C(K)$ to be finite.

• By Kodaira-Parshin construction,

up to replacing C by a finite étale cover,

can assume $\exists \pi: X \rightarrow C$ family of curves of genus $g > 1$.

$\exists S \subset M_K$ finite s.t. $\forall s \in C(\bar{K})$, X_s/K has good reduction outside S .

• Consider $C(K) \xrightarrow{\mathbb{I}} \{G_K\text{-rep'n } / \mathbb{Q}_p \text{ of dim } 2g, \text{ pure of wt } 1 \text{ outside } S\}$

$$\begin{array}{ccc} C(K) & \xrightarrow{\mathbb{I}} & \{G_K\text{-rep'n } / \mathbb{Q}_p \text{ of dim } 2g, \text{ pure of wt } 1 \text{ outside } S\} \\ \downarrow \text{ir} & \searrow & \downarrow \text{jr} \\ C(K_v) & \xrightarrow{\mathbb{I}_v} & \{G_{K_v}\text{-rep'n } / \mathbb{Q}_p \text{ of dim } 2g\} \end{array}$$

Choose v place of K above p , $v \in S$ (can enlarge S if necessary)

$\hookrightarrow X_s/K$ good red at v and v unram above p .

Key thm (LV) \mathbb{I}_v is finite-to-one

\hookrightarrow Assuming key thm, the Mordell is as follows:

\mathbb{I}_v is finite-to-one

$\Rightarrow \mathbb{I}_v \circ \text{ir} = \text{jr} \circ \mathbb{I}$ is finite-to-one

$\Rightarrow \mathbb{I}$ is finite-to-one.

By Faltings' thm, $\#\{G_K\text{-rep'n}\}$ finite.

Then $C(K)$ is finite.

Remark (i) Faltings' thm says the G_K -rep'n

$$V_p(A) = H^1(A/\bar{K}, \mathbb{Q}_p)^\vee \text{ is semisimple.}$$

But LV doesn't assume it.

In fact, LV proved the semisimplicity for all but finitely many $s \in C(K)$.

(2) \mathbb{I}_v is purely local.

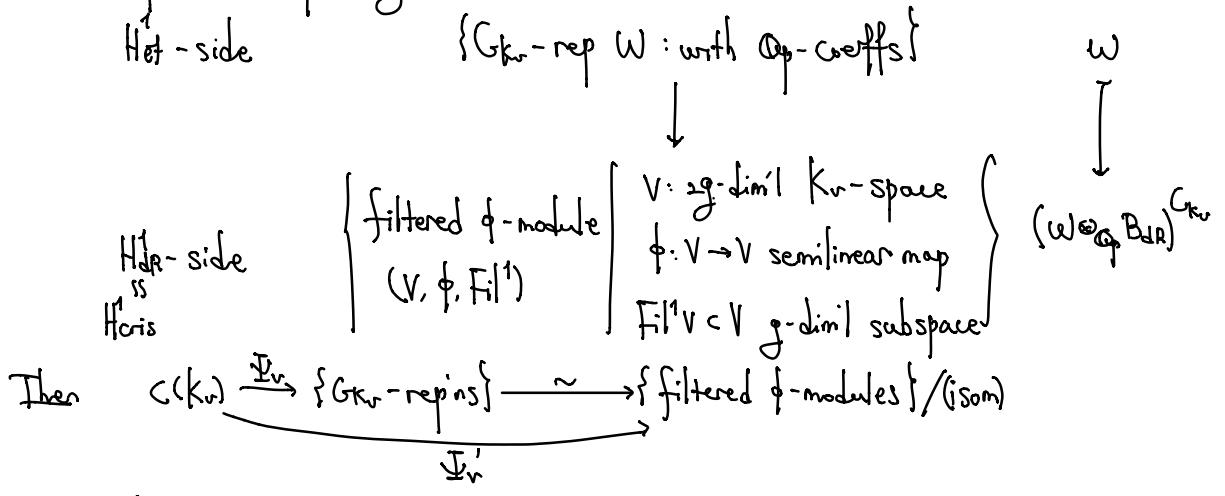
Recall p -adic Hodge theory:

(1) (Comparison) Y/K_v curve ($Y = Y_s$)

$$H_{\text{ét}}^1(\bar{Y}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H_{\text{dR}}^1(Y/K_v) \otimes_{K_v} B_{\text{dR}}$$

$$H_{\text{ét}}^1(\bar{Y}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H_{\text{cris}}^1(Y_{K_v}/K_v) \otimes_{K_v} B_{\text{cris}}$$

(2) (Equivalence of categories)

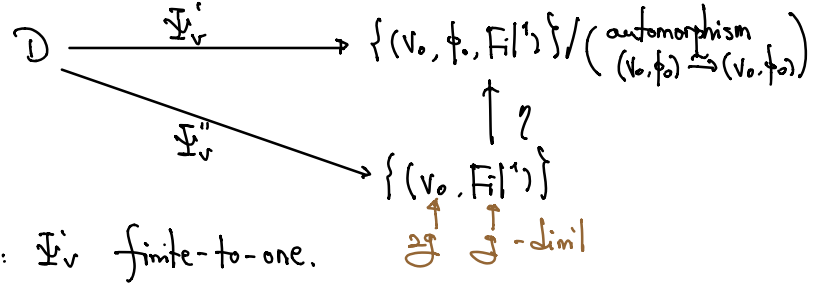


take a residue disc $D \subset C(K_v)$

(all $s \in D$ give fiber Y_s with the same reduction).

\Rightarrow it trivializes crystalline cohomology with ϕ -action.

Can assume $(V, \phi) = (V_0, \phi_0)$.



Need: Ψ_v' finite-to-one.

Two steps (1) $\Psi_v'' : D \rightarrow \text{Gr}(K_v)$ ("analytic") is finite-to-one.
 \uparrow \uparrow
 p -adic manifold algebraic variety
 ($\text{Gr} = \text{Grassmannian}/K_v$).

(2) $(\text{im } \Psi_v'') \cap (\text{fiber of } \gamma)$ finite, for any fiber.

fibers of ρ are orbits of centralizer $Z(\phi)$ of ϕ in $GL(V_0)$.
Need $Z(\phi)$ "small".