LV Seminar

Fibers with good reduction in a family

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Notation

- K: a number field;
- $\overline{K}$: a fixed algebraic closure of K;
- $G_K = \mathrm{Gal}_{\overline{K}/K}$: the absolute Galois group;
- S: a finite set of places of K containing all the archimedean places;
- $\mathcal{O}_S$: the ring of S-integers ($|x|_v \leqslant 1$ for all $v \notin S$), abbreviated to $\mathcal{O}$ in Lawrence and Venkatesh's paper (and this talk);
- p: a (rational) prime number such that no place of S lies above p;
- K_w: the completion of K at a prime w of $\mathcal{O} = \mathcal{O}_S$ (i.e. $w \notin S$);
- $\overline{K}_w$: a fixed algebraic closure of K_w;
- $\mathbf{F}_w$: the residue field at w;
- q_w: the cardinality of $\mathbf{F}_w$;
- $\overline{\mathbf{F}}_w$: the residue field of $\overline{K}_w$, which is an algebraic closure of F_w;
- $\mathcal{O}_{(w)}$: the localization of $\mathcal{O} = \mathcal{O}_S$ at w.

Notation

A G_K-set means a (discretely topologized) set with a continuous action of G_K.
For a variety X over a field E of characteristic zero, we denote by $H^*_{\mathrm{dR}}(X/E)$ the de Rham cohomology of $X \to \operatorname{Spec} E$. If $E' \supset E$ is a field extension, we denote by $H^*_{\mathrm{dR}}(X/E')$ the de Rham cohomology of the base-change $X_{E'}$, which is identified with $H^*_{\mathrm{dR}}(X/E) \otimes_E E'$.
For any scheme S, a family over S is an (arbitrary) S-scheme $\pi\colon Y \to S$. A curve over S is a family over S such that π is smooth and proper of relative dimension 1 and each geometric fiber is connected.

Notation

Let $E/\mathbf{Q}_p$ be a finite unramified extension of $\mathbf{Q}_p$, and σ the unique automorphism of E lifting the (usual or arithmetic) Frobenius map on the residue field. A ϕ-module (over E) means a pair (V, ϕ), with V a finite-dimensional E-vector space and $\phi \colon V \to V$ a map semilinear with respect to σ. A filtered ϕ-module will be a triple $(V, \phi, F^i V)$ such that (V, ϕ) is a ϕ-module and $(F^i V)_i$ is a descending filtration on V. We demand that each $F^i V$ be an E-linear subspace of V but require no compatibility with ϕ. Note that the filtered ϕ-modules arising from Galois representations via p-adic Hodge theory satisfy a further condition, admissibility, but we do not need it.

Lemma 1 (Lemma 2.3, Faltings)

Fix integers $w, d \geqslant 0$, and fix K and S as above. There are, up to conjugation, only finitely many semisimple Galois representations $\rho\colon G_K \to \mathrm{GL}_d(\mathbf{Q}_p)$ such that

(a) ρ is unramified outside S, and

(b) ρ is pure of weight w, i.e. for every prime $\wp \notin S$ all roots of the characteristic polynomial of Frobenius at $\wp$ are algebraic with complex absolute value $q_\wp^{w/2}$.

(c) For $\wp$ as above the characteristic polynomial of Frobenius at $\wp$ has integer coefficients.

In Section 3, Lawrence and Venkatesh give a general criterion (Proposition 3.4) which controls, in a given family of smooth proper varieties, the subset of base scheme whose fibers have good reduction outside a fixed set of primes. The Proposition simply translates (using p-adic Hodge theory) the finiteness statement of Lemma 2.3 into a restriction on the image of the period map (will be introduced later).

Remark In our settings, "good reduction" means the Galois representation on the étale cohomology of the special fiber is semisimple. We will assume a good model of $X \to Y$ from the very beginning, so all the fibers are smooth proper schemes.

Notation for Section 3

Let Y be a smooth K-variety, and $\pi\colon X \to Y$ be a smooth proper morphism. Assume that π admits a good model over $\mathcal{O}$ $(= \mathcal{O}_S)$, i.e. it extends to a smooth proper morphism $\pi\colon \mathcal{X} \to \mathcal{Y}$ of smooth $\mathcal{O}$-schemes. Assume further that all the cohomology sheaves $\mathbf{R}^q\pi_*\Omega^p_{\mathcal{X}/\mathcal{Y}}$ are sheaves of locally free $\mathcal{O}_Y$-modules, and that the same is true for the relative de Rham cohomology $\mathscr{H}^q = \mathbf{R}^q\pi_*\Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$. There is no harm in these assumptions, because the sheaves in question are coherent $\mathcal{O}_Y$-modules which are free at the generic point of $\mathcal{O}$; so the assumptions can always be achieved by possibly enlarging the set S of primes.

The generic fiber of $\mathscr{H}^q$ is equipped with the Gauss–Manin connection and, again by enlarging S if necessary, we may assume that this extends to a morphism

$$\mathscr{H}^q \to \mathscr{H}^q \otimes \Omega^1_{\mathcal{Y}/\mathcal{O}}. \tag{1}$$

Notation for Section 3

For any $y \in Y(K)$, we denote by $X_y = \pi^{-1}(y)$ the fiber of π above y, which is a smooth proper variety over K. Our goal is to bound $Y(\mathcal{O})$. We will do this by studying the p-adic properties of the Galois representation attached to X_y, for $y \in \mathcal{Y}(\mathcal{O}) \hookrightarrow Y(K)$. Fixing a degree $q \geqslant 0$, we denote by ρ_y the representation of the Galois group G_K on the étale cohomology group of $(X_y)_{\overline{K}}$:

$$\rho_y \colon G_K \to \operatorname{Aut}\left(H^q_{\mathrm{et}}(X_y \times_K \overline{K}, \mathbf{Q}_p)\right).$$

Notation for Section 3

Fix an archimedean place $\iota\colon K \hookrightarrow \mathbf{C}$, and fix a finite place $v\colon K \hookrightarrow K_v$ satisfying:

- if p is the rational prime below v, then $p > 2$, and
- K_v is unramified over $\mathbf{Q}_p$, and
- no prime above p lies in S.

Fix $y_0 \in \mathcal{Y}(\mathcal{O})$, and let

$$U := \{y \in \mathcal{Y}(\mathcal{O}) \mid y \equiv y_0 \mod v\}.$$

Proposition 3.4 will give criteria for the finiteness of U in terms of the associated period map. If U is finite for each y_0, then $\mathcal{Y}(\mathcal{O})$ is also finite.

Finally, let $X_0 = \pi^{-1}(y_0(\operatorname{Spec} K))$ be the generic fiber above $y_0(\operatorname{Spec} K) \in Y$.

Cohomology at the Basepoint y_0

For any K-variety Z, we shall denote by $Z_{\mathbf{C}}$ its base change to $\mathbf{C}$ via ι, and by Z_{K_v} its base change to K_v via v.
Let $V = H^q_{dR}(X_0/K)$, and let V_v (resp. $V_{\mathbf{C}}$) denote the K_v- (resp. $\mathbf{C}$-) vector spaces obtained by $\otimes_K K_v$ (resp. $\otimes_{(K,\iota)}\mathbf{C}$). Then $V_{\mathbf{C}}$ is naturally identified with the de Rham cohomology of the variety $X_{0,\mathbf{C}}$, which is also (by the comparison theorem) identified with the singular cohomology of $X_{0,\mathbf{C}}$ with complex coefficients, i.e.

$$V_{\mathbf{C}} \simeq H^q_{\text{sing}}(X_{0,\mathbf{C}}, \mathbf{C}).$$

In particular, monodromy defines a representation $\mu\colon \pi_1(Y_{\mathbf{C}}(\mathbf{C}), y_0) \to \mathrm{GL}(V_{\mathbf{C}})$. Let Γ be the Zariski closure of $\mathrm{im}(\mu)$, which is an algebraic subgroup of $\mathrm{GL}(V_{\mathbf{C}})$. Although both $V_{\mathbf{C}}$ and Γ depend on the choice of the archimedean place ι, we will ignore this dependence from our notation.

Gauss–Manin Connection

The connection (1) allows us to indentify the cohomology of nearby fibers. This is true both for the K_v and $\mathbf{C}$ topologies. We will show that both identifications are locally given by the same power series with K coefficient, which is convergent both in the K_v and $\mathbf{C}$ topologies.

Specifically, let $\{v_1, \ldots, v_r\}$ be a local basis for $\mathscr{H}^q$ in a neighborhood of some point on $\mathcal{Y}$. Write $\nabla v_i = \sum_j A_{ij} v_j$, where A_{ij} are local sections of $\Omega^1_{\mathcal{Y}}$. A (local) section f of $\mathscr{H}^q$ is called *flat* if

$$\nabla f = 0.$$

Gauss–Manin Connection

Let $f = \sum_i f_i v_i$. Then

$$\nabla f = \nabla\left(\sum_i f_i v_i\right) = \sum_i (\nabla f_i v_i) = \sum_i \left(\mathrm{d}(f_i) v_i + f_i \nabla v_i\right)$$
$$= \sum_i \left(\mathrm{d}(f_i) v_i + f_i \sum_j A_{ij} v_j\right) = \sum_i \mathrm{d}(f_i) v_i + \sum_{ij} f_i A_{ij} v_j$$
$$= \sum_i \mathrm{d}(f_i) v_i + \sum_{ij} f_j A_{ji} v_i = \sum_i \left(\mathrm{d}(f_i) + \sum_j f_j A_{ji}\right) v_i.$$

Thus f is flat $\iff$

$$\mathrm{d}(f_i) = -\sum_j A_{ji} f_j. \tag{2}$$

Gauss–Manin Connection

In particular, if $y_0 \in \mathcal{Y}(\mathcal{O})$ and the place v is as before, let $\bar{y}_0 \in \mathcal{Y}(\mathbf{F}_v)$ be the reduction. Since K_v is unramified over $\mathbf{Q}_p$, p generates the maximal ideal of the local ring $\mathcal{O}_{(v)}$, and we can choose a system of parameters (local coordinates) $p, z_1, \ldots, z_m \in \mathcal{O}_{\mathcal{Y},\bar{y}_0}$ such that $z_1, \cdots, z_m$ generate the kernel of $\mathcal{O}_{\mathcal{Y},\bar{y}_0} \to \mathcal{O}_{(v)}$ (induced by $y_0 \colon \operatorname{Spec} \mathcal{O} \to \mathcal{Y}$). Then $\hat{\mathcal{O}}_{\mathcal{Y},\bar{y}_0} = \mathcal{O}_v[[z_1, \ldots, z_m]]$ and $\mathcal{O}_{\mathcal{Y},\bar{y}_0} \subseteq \mathcal{O}_{(v)}[[z_1, \ldots, z_m]]$.

Fix a basis $\{\bar{v}_1, \ldots, \bar{v}_m\}$ for $\mathscr{H}^q$ at $\bar{y}_0$ such that each step of the Hodge filtration $F^i\mathscr{H}^q$ at $\bar{y}_0$ is spanned by a subset of $\{\bar{y}_i\}$. Then by lifting we obtain a similar basis $\{v_1, \ldots, v_r\}$ for $\mathscr{H}^q$ at y_0 over $\mathcal{O}_{Y,y_0}$. With respect to the basis $\{v_i\}$, the coefficients A_{ij} of (2) are of the form $A_{ij} = \sum_{k=1}^m a_{ij,k} \mathrm{d}z_k$, where $a_{ij,k} \in \mathcal{O}_{Y,\bar{y}_0} \subseteq \mathcal{O}_{(v)}[[z_1, \ldots, z_m]]$.

Gauss–Manin Connection

Write a formal solution (f_i) to (2) as formal power series in $K[[z_1, \ldots, z_m]]$. We see that these formal power series are v-adically absolute convergent for $|z_i|_v < |p|_v^{1/(p-1)}$ ($p = \mathrm{Char}\, \mathbf{F}_v$) and ι-adically absolute convergent for sufficiently small $|z_i|_{\mathbf{C}}$. (For archimedean place, this follows from usual linear ODE theory; for nonarchimedean place, this follows from strong triangle inequality.)

Example

Suppose that $m = r = 1$. Then

$$\mathrm{d}(f) = af\mathrm{d}z \implies f = \text{some constant} \cdot \exp\left(\int a\mathrm{d}z\right).$$

Let

$$a = a_k z^k + \text{higher degree terms} \quad (a_k \in \mathcal{O}_{(v)}).$$

Then

$$\int a\mathrm{d}z = \frac{a_k}{k+1} z^{k+1} + \text{higher degree terms.}$$

It is easy to show that $|z|^k < |p|_v^{k/(p-1)} \leqslant |k+1|_v$ (hint: consider $|(k+1)!|_v$). Therefore, for $|z|_v < |p|_v^{1/(p-1)}$,

$$\left|\int a\mathrm{d}z\right|_v \leqslant \left|\frac{a_k}{k+1} z^{k+1}\right|_v \leqslant |z|_v < |p|_v^{1/(p-1)}$$

and thus $\exp\left(\int a\mathrm{d}z\right)$ converges absolutely.

v-adic neighborhood of y_0

We will often consider the following v-adic neighborhood of y_0

$$\Omega_v \coloneqq \{y \in \mathcal{Y}(\mathcal{O}_v) \mid y \equiv y_0 \mod v\}.$$

So we clarify it a bit.
$\operatorname{Spec} \mathcal{O}_v = \{(0), \mathfrak{m}_v\}$. So the image of $\mathcal{Y}(\mathcal{O}_v) \ni y \colon \operatorname{Spec} \mathcal{O}_v \to \mathcal{Y}$ consists of 2 points, one on the generic fiber Y and the other one on the special fiber above v. If we write X_y, we mean the preimage of $y((0))$ under $X \to Y$.
$y \equiv y_0 \mod v$ means that $y(\mathfrak{m}_v) = y_0(\mathfrak{m}_v)$, i.e. $X_{y,\mathbf{F}_v} = X_{y_0,\mathbf{F}_v}$ over the residul field $\mathbf{F}_v$. In other words, X_y and X_{y_0} (both are K_v-varieties) have the same reduction.
We have natural inclusions $\mathcal{Y}(\mathcal{O}) \subseteq \mathcal{Y}(\mathcal{O}_v)$, $U \subseteq \Omega_v$ (definition).
Via the first inclusion we have $X_{y_0} = X_{0,v}$ (definition, base-change notation).

Gauss–Manin Connection

Since $p > 2$ and v is unramified over p, we have identifications

$$\mathrm{GM}\colon H^q_{\mathrm{dR}}(X_{y_0}/K_v) \xrightarrow{\simeq} H^q_{\mathrm{dR}}(X_y/K_v)$$

for all $y \in \Omega_v$, and

$$\mathrm{GM}\colon H^q_{\mathrm{dR}}(X_{y_0,\mathbf{C}}/\mathbf{C}) \xrightarrow{\simeq} H^q_{\mathrm{dR}}(X_{y,\mathbf{C}}/\mathbf{C})$$

for all $y \in Y_{\mathbf{C}}(\mathbf{C})$ sufficient close to y_0. In terms of the basis v_i and local coordinates $z_1, \ldots, z_m$ chosen above, the two GM's are given by the same an $r \times r$ matrix with entries

$$A_{ij} \in \mathcal{O}_{(v)}[[z_1, \ldots, z_m]]$$

convergent in the neighborhoods of y_0 noted above (so we can abuse the notation and call both of them GM).

Gauss–Manin Connection

The fiber over the $\mathcal{O}$-point y_0 of $\mathcal{Y}$ provides a smooth proper $\mathcal{O}$-model $\mathcal{X}_0$ for X_0. For $y \in \Omega_v$, by comparison theorem we have a commutative diagram

$$\begin{array}{ccc} H^q_{\mathrm{dR}}(X_{y_0}/K_v) & \overset{\simeq}{\searrow} & \\ \Big\downarrow \mathrm{GM} & & H^q_{\mathrm{cris}}(\bar{\mathcal{X}}_0) \otimes_{\mathcal{O}_v} K_v, \\ H^q_{\mathrm{dR}}(X_y/K_v) & \overset{\simeq}{\nearrow} & \end{array} \tag{3}$$

where the diagonal arrows are canonical identifications.

The crystalline cohomology $H^q_{\mathrm{cris}}(\bar{\mathcal{X}}_0)$ (where $\bar{\mathcal{X}}_0$ is the "completion" of $\mathcal{X}_0$ with respect to v) is equipped with a Frobenius operator $\phi_v\colon H^q_{\mathrm{cris}}(\bar{\mathcal{X}}_0) \to H^q_{\mathrm{cris}}(\bar{\mathcal{X}}_0)$ which is semilinear with respect to the Frobenius on the unramified extension $K_v/\mathbf{Q}_p$. By the above identifications, ϕ_v acts on $H^q_{\mathrm{dR}}(X_{y_0}/K_v)$ and $H^q_{\mathrm{dR}}(X_y/K_v)$ as well, and these actions are compatible with the map GM.

Period map in a neighborhood of y

$V = H^q_{\mathrm{dR}}(X_0/K)$ is equipped with a Hodge filtration

$$V = F^0 V \supseteq F^1 V \supseteq \cdots \tag{4}$$

Let $\mathcal{H}$ be the K-variety parameterizing flags in V with the same dimensional data as (4), and $h_0 \in \mathcal{H}(K)$ be the point corresponding to the Hodge filtration on V.

Remark Recall that $X_{y_0} = X_{0,v}$. So

$$H^q_{\mathrm{dR}}(X_{y_0}/K_v) = H^q_{\mathrm{dR}}(X_0/K) \otimes_K K_v = V \otimes_K K_v.$$

I think this is why Lawrence and Venkatesh let V_v denote $H^q_{\mathrm{dR}}(X_{y_0}/K_v)$.

Period map in a neighborhood of y

Recall that we have a K_v-variety $\mathcal{H}_v$ and a $\mathbf{C}$-variety $\mathcal{H}_{\mathbf{C}}$. Let $h_0^\iota \in \mathcal{H}_{\mathbf{C}}(\mathbf{C})$ denote the base-change of h_0, and let $\Omega_{\mathbf{C}}$ be a contractible analytic neighborhood of $y_0 \in Y_{\mathbf{C}}^{\mathrm{an}}$. The Gauss-Manin connection defines an isomorphism $H^q_{\mathrm{dR}}(X_t/\mathbf{C}) \simeq H^q_{\mathrm{dR}}(X_0/\mathbf{C})$ for each $t \in \Omega_{\mathbf{C}}$. In particular, the Hodge filtration on $H^q_{\mathrm{dR}}(X_t/\mathbf{C})$ defines a point in $\mathcal{H}_{\mathbf{C}}(\mathbf{C})$, and thus gives rise to the complex period map

$$\Phi_{\mathbf{C}} \colon \Omega_{\mathbf{C}} \to \mathcal{H}_{\mathbf{C}}(\mathbf{C}). \tag{5}$$

Moreover, $\Phi_{\mathbf{C}}$ extends to a map from $\widetilde{Y}_{\mathbf{C}}^{\mathrm{an}}$, the universal covering space of $Y_{\mathbf{C}}^{\mathrm{an}}$, to $\mathcal{H}_{\mathbf{C}}(\mathbf{C})$ such that the map is equivariant under the monodromy action of $\pi_1(Y_{\mathbf{C}}^{\mathrm{an}}, y_0)$ on $\mathcal{H}_{\mathbf{C}}(\mathbf{C})$. The following lemma shows that the image of the period map can be bounded below by monodromy.

Period map in a neighborhood of y

Lemma 2 (Lemma 3.1)

Suppose given a family $X \to Y$, and take notations as before; in particular, Γ is the Zariski closure of monodromy, and $h_0^\iota = \Phi_{\mathbf{C}}(y_0)$. Then we have the following inclusion

$$\Gamma \cdot h_0^\iota \subseteq \text{the Zariski closure of } \Phi_{\mathbf{C}}(\Omega_{\mathbf{C}}) \text{ in } \mathcal{H}_{\mathbf{C}}.$$

Proof.

For any algebraic subvariety $Z \subseteq H_{\mathbf{C}}$ such that $Z \supseteq \Phi_{\mathbf{C}}(\Omega_{\mathbf{C}})$, $\Phi_{\mathbf{C}}^{-1}(Z)$ is a complex-analytic subvariety of $\widetilde{Y}_{\mathbf{C}}^{\mathrm{an}}$ containing (a complex topology neighborhood) $\Omega_{\mathbf{C}}$, and thus $\Phi_{\mathbf{C}}^{-1}(Z) = \widetilde{Y}_{\mathbf{C}}^{\mathrm{an}}$. Therefore, $\pi_1(Y_{\mathbf{C}}, y_0) \cdot h_0^\iota \subseteq Z$. Then we deduce that Z contains the Zariski closure of $\pi_1(Y_{\mathbf{C}}, y_0) \cdot h_0^\iota$, i.e. $\Gamma \cdot h_0^\iota$. □

Period map in a neighborhood of y

We need a v-adic analogue of the above lemma. For each $y \in \Omega_v$, the Gauss–Manin connection (3) allows us to identify the Hodge filtration on $H^q_{\mathrm{dR}}(X_y/K_v)$ with a filtration on $V_v = H^q_{\mathrm{dR}}(X_{y_0}/K_v)$, and thus with a point in $\mathcal{H}(K_v)$. This gives rise to a K_v-analytic map

$$\Phi_v \colon \Omega_v \to \mathcal{H}_v(K_v). \tag{6}$$

The following simple lemma plays a crucial role. It allows us to analyze the Zariski closure of the p-adic period map in terms of the Zariski closure of the complex period map. Recall that the latter is bounded below by monodromy.

Period map in a neighborhood of y

Lemma 3 (Lemma 3.2)

Suppose given power series $B_0, \ldots, B_N \in K[[z_1, \ldots, z_m]]$ such that all B_i converge absolutely, with no common zero, both in the v-adic disk

$$U_v = \{(z_1, \ldots, z_m) \mid |z_i|_v < \epsilon, \ \forall i\}$$

and the complex disk

$$U_{\mathbf{C}} = \{(z_1, \ldots, z_m) \mid |z_i|_{\mathbf{C}} < \epsilon, \ \forall i\}.$$

Write

$$\underline{B}_v \colon U_v \to \mathbf{P}^N_{K_v}, \quad \underline{B}_{\mathbf{C}} \colon U_{\mathbf{C}} \to \mathbf{P}^N_{\mathbf{C}},$$

for the corresponding maps.
Then there exists a K-subscheme $\mathcal{Z} \subseteq \mathbf{P}^N$ whose base extension to K_v (resp. $\mathbf{C}$) gives the Zariski closure of $\underline{B}_v(U_v) \subseteq \mathbf{P}^N_{K_v}$ (resp. $\underline{B}_{\mathbf{C}}(U_{\mathbf{C}}) \subseteq \mathbf{P}^N_{\mathbf{C}}$). In particular, these Zariski closures have the same dimension.

Proof of Lemma 3.

Let I be the ideal that generated by all homogeneou polynomials $Q \in K[x_0, \ldots, x_N]$ such that $Q(B_0, \ldots, B_N) \equiv 0$, and let $\mathcal{Z}$ be the subscheme of $\mathbf{P}^N$ defined by I.

To verify the claim for K_v (the proof for $\mathbf{C}$ can be obtained by replacing all the v and K_v by $\mathbf{C}$ in the following argument), it suffices to verify that if a homogeneous polynomial $Q_v \in K_v[x_0, \ldots, x_N]$ vanishes on $B_v(U_v)$ then Q_v lies in the K_v-span of I. For $Q_v \in K_v[x_0, \ldots, x_N]$ vanishing on $B_v(U_v)$, $Q(B_0, \ldots, B_N) \equiv 0$ in $K_v[[z_1, \ldots, z_m]]$, which is equivalent to the fact that the coefficients of Q_v satisfy certain infinite system of linear equations with coefficients in K. Since Q_v is a polynomial, there are only finite variables in the linear equation system. So any K_v-solution of such a linear system is a K_v-linear combination of K-solutions. □

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Remark I think the proof reveals half of the essence of GAGA (the other half is to show a compact analytic variety is Zariski closed).

By embedding $\mathcal{H}$ into a projective space $\mathbf{P}^N$, we can deduce that:

Lemma 4 (Lemma 3.3)

The dimension of the Zariski closure (in the K_v-variety $\mathcal{H}_v$) of $\Phi_v(\Omega_v)$ is at least the (complex) dimension of $\Gamma \cdot h_0^\iota$.
In particular, if $\mathcal{H}_v^{\mathrm{bad}} \subset \mathcal{H}_v$ is a Zariski-closed subset of dimension less than $\dim_{\mathbf{C}}(\Gamma \cdot h_0^\iota)$, then $\Phi_v^{-1}(\mathcal{H}_v^{bad})$ is contained in a proper K_v-analytic subset of Ω_v, by which we mean a subset cut out by v-adic power series converging absolutely on Ω_v.

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We can improve this result using the results of Bakker and Tsimerman, replacing "proper *Kv*-analytic" by "Zariski-closed". See Section 9 of Lawrence and Venkatesh's paper. We do no need this improvement for the applications to Mordell conjecture.

Proof of Lemma 4.

Since Φ_v and $\Phi_{\mathbf{C}}$ are both induced by GM, they are given by the same power series with coefficients in K. By Lemma 3,

$$\dim \text{of Zariski closure of } \Phi_v(\Omega_v) = \dim \text{of Zariski closure of } \Phi_{\mathbf{C}}(\Omega_{\mathbf{C}}).$$

By Lemma 2,

$$\dim_{\mathbf{C}}(\Gamma \cdot h_0^{\iota}) \leqslant \dim \text{of Zariski closure of } \Phi_{\mathbf{C}}(\Omega_{\mathbf{C}}).$$

Therefore,

$$\dim_{\mathbf{C}}(\Gamma \cdot h_0^{\iota}) \leqslant \dim \text{of Zariski closure of } \Phi_v(\Omega_v).$$

□

Hodge Structure

We use p-adic Hodge theory to relate Galois representations to crystalline cohomology. For each $y \in U$, the representation ρ_y (definition) restricting to G_{K_v} for a chosen homomorphism $G_{K_v} \to G_K$ is crystalline, because of the existence of the model $\mathcal{X}_y$ for X_y. By p-adic Hodge theory, there is a fully faithful embedding of categories: (The essential image are weakly admissible objects by Hu's last lecture, but we do not need this description.)

$$\text{crystalline representations of } \mathrm{Gal}_{K_v} \text{ on } \mathbf{Q}_p\text{-vector spaces} \hookrightarrow \mathcal{FL},$$

where the objects of $\mathcal{FL}$ are triples (W, ϕ, F) consisting of a K_v-vector space W, a Frobenius-semilinear automorphism $\phi\colon W \to W$, and a descending filtration F on W. The morphisms in the category $\mathcal{FL}$ are morphisms of K_v-vector spaces that are compatible with ϕ and filtrations.

Hodge Structure

By the crystalline comparison theorem of Faltings, the above embedding takes ρ_y to the triple $(H^q_{\mathrm{dR}}(X_y/K_v), \phi_v, \text{Hodge filtration for } X_y)$. And (6) induces an isomorphism in $\mathcal{FL}$:

$$(H^q_{\mathrm{dR}}(X_y/K_v), \phi_v, \text{Hodge filtration for } X_y) \simeq (V_v, \phi_v, \Phi_v(y)).$$

As a sample result of what we can show now, we give the following proposition. We will use the method of its proof again and again, so it seems useful to present the method in the current simple context.

Proposition 5 (Proposition 3.4)

Notations as above. In particular, $\pi\colon X \to Y$ *is a smooth proper family over* K, $y_0 \in Y(K)$, $X_0 = \pi^{-1}(y_0)$, $V = H^q_{\mathrm{dR}}(X_0/K)$, $\mathcal{H}$ *the space of flags in* V,

$$\Phi_v\colon \Omega_v = \{y \in \mathcal{Y}(\mathcal{O}_v) \mid y \equiv y_0 \mod v\} \to \mathcal{H}(K_v)$$

is the v*-adic period map,* $\Gamma \subseteq \mathrm{GL}(V_{\mathbf{C}})$ *is the Zariski closure of the monodromy group, and* $h_0 = \Phi(y_0)$.
Suppose that

$$\dim_{K_v}(Z\left(\phi_v^{[K_v:\mathbf{Q}_p]}\right) < \dim_{\mathbf{C}} \Gamma \cdot h_0^{\iota}, \tag{7}$$

where on the left $Z(\dots)$ *denotes the centralizer of the* K_v*-linear operator* $\phi_v^{[K_v:\mathbf{Q}_p]}$ *(i.e. commuting with it) in* $\mathrm{GL}_{K_v}(V_v)$. *Then the set*

$$\{y \in \mathcal{Y}(\mathcal{O}) \mid y \equiv y_0 \mod v,\ \rho_y \text{ is semisimple}\} \tag{8}$$

is contained in a proper K_v*-analytic subvariety of* Ω_v.

Proof of Proposition 5.

Note that $\mathrm{GL}_{K_v}(V_v)$ acts on flags in V_v, so $\mathrm{GL}_{K_v}(V_v)$ acts on $\mathcal{H}_v$.

For any y as in (8), the Galois representation ρ_y belongs to a finite set of isomorphism classes by Lemma 1. By our previous discussion, the triple $(V_v, \phi_v, \phi_v(y))$ also belongs to a finite set of isomorphism classes in the category $\mathcal{FL}$. Choosing representatives (V_v, ϕ_v, h_i) for these isomorphism classes. There exists some h_{i_0} such that $\Phi_v(y)$ differs from (V_v, ϕ_v, h_{i_0}) by an isomorphism in $\mathcal{FL}$, which is an element in $\mathrm{GL}_{K_v}(V_v)$ commuting with ϕ_v. Thus

$$\Phi_v(y) \in \bigcup_i Z(\phi_v) \cdot h_i,$$

where $Z(\phi_v)$ is the subgroup of elements in $\mathrm{GL}_{K_v}(V_v)$ which commute with ϕ_v.

Proof of Proposition 5 (Cont.)

$$\Phi_v(y) \in \bigcup_i Z(\phi_v) \cdot hi$$

Now certainly $Z(\phi_v) \subseteq Z(\phi_v^{[K_v:\mathbf{Q}_p]})$, and the right side is the K_v-points of a K_v-algebraic subgroup of $\mathrm{GL}_{K_v}(V_v)$. Therefore, the set (8) is contained in the preimage, under ϕ_v, of a proper Zariski closed subset of $\mathcal{H}_v$ with dimension $\dim_{K_v}(Z(\phi_v^{[K_v:\mathbf{Q}_p]})$. This is obviously a K_v-analytic subvariety as asserted. It is proper by (7) and Lemma 4. □

In conclusion we note that instead of bounding $Y(K)$, we bound $\mathcal{Y}(\mathcal{O}) = \mathcal{Y}(\mathcal{O}_S)$ where the Galois representation ρ_y ($y \in \mathcal{Y}(\mathcal{O})$) has good reduction outside S. To bound $Y(K)$, one would have to deal y that are perhaps non-integral at S. This would require a more detailed analysis "at infinity", and Lawrence and Venkatesh have not attempted it.