Lv. §6 (Rationality of ab-fin family)

<u>Prop 5.3</u> Suppose $X \to Y' \to Y$ has good model $/ \mathcal{O} = \mathcal{O}_{K,S} = \mathcal{O}_K[\frac{1}{S}]$.
and $v \notin S$ friendly place of K.

$Y(K)^* = \{ y \in Y(K) : \mathrm{size}_v(\pi^{-1}(y)) < \frac{1}{d+1} \}$ is finite. $d = $ rel dim X/Y'.

if E = finite set, then

$G_K \circlearrowright E$ $\quad \mathrm{size}_v(E) = \dfrac{\#\{x \in E \text{ s.t. Frob-orbit of } x \text{ is } < 8\}}{\# E}$

= proportion of pts w/ large ext'n $K(y')/K(y)$.

<u>§1 Basic structures in ab-fin family</u>

- $y_0 \in Y(K)^*$, $\Omega_v = \{ y \in Y(K_v) : y \equiv y_0 \bmod v \}$.

  $\rightsquigarrow$ to prove $\Omega_v \cap \boxed{Y(K)^*}$ finite

  covered by finitely many v-adic discs

- $y \in Y(K) \cap \Omega_v$.

  $\pi^{-1}(y) = \mathrm{Spec}\, E_y$

  $E_y = \Gamma(Y' \times_Y y, \mathcal{O})$.

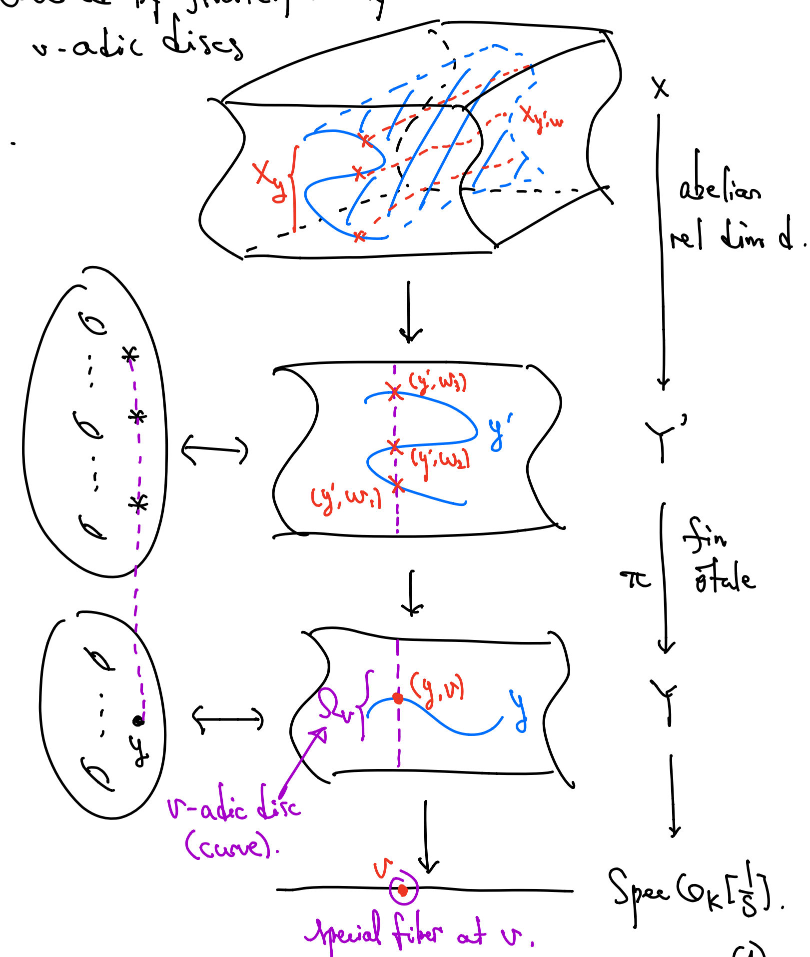
  fin étale K-alg.

  X_y / E_y sch.

  $= \prod_{y'|y} K(y')$.

  $y' \in Y'(\bar{K})$ s.t. $\pi(y') = y$

  $X_y / E_y = \coprod_{y'|y} X_{y'}$.



- Need to decompose every y' (y does not carry enough info yet.)

$$\begin{tikzcd}
 & \coprod X_{y'} \arrow[d, phantom, "\shortparallel"] & \\
X_{y'} \arrow[r] \arrow[d] & X_y \arrow[r] \arrow[d] & X \arrow[d, "\substack{\text{polarized AS}\\ \text{of rel dim } d}"] \\
\operatorname{Spec} k(y') \arrow[r, "y'"] & \operatorname{Spec} E_y \arrow[r] \arrow[d, phantom, "\shortparallel"] & Y' \arrow[dd, "\pi,\ \text{fin ét.}"] \\
 & \operatorname{Spec}(\prod k(y')) \arrow[d] & \\
 & \operatorname{Spec} K = y \arrow[r] & Y
\end{tikzcd}$$

$\rho_{y'} : \mathrm{Gal}_K \to GL(V)$ ($\rho_{y'}$) $\rightsquigarrow$

$\rightsquigarrow \rho_{y'}|\mathrm{Gal}_{K_v}$.

⚠ $\boxed{\text{Simple or not ?}}$

we truly concern about.

Algebraic structures

(1) E_y/K fin étale $\Rightarrow$ E_y unram, $\Omega^1_{E_y/K} = 0$.

$$\Rightarrow H^i_{dR}(X_y/K) = H^i_{dR}(X_y/E_y) = \bigoplus_{y'|y} H^i_{dR}(X_{y'}/k(y')).$$

$$\circlearrowleft \qquad \Uparrow$$

$$\mathcal{O}_{E_y} = \prod \mathcal{O}_{k(y')}$$

(2) X/Y' polarization $X \to X^\vee$ $\rightsquigarrow$ symplectic E_y-bilin pairing

$$\omega : H^1_{dR}(X_y/E_y) \times H^1_{dR}(X_y/E_y) \longrightarrow E_y,$$

$$\shortparallel$$

$$\bigoplus \omega_{y'} : H^1_{dR}(X_{y'}/k(y')) \times H^1_{dR}(X_{y'}/k(y')) \longrightarrow E_y,$$

(3) Set $E_{y,v} = E_y \otimes_K K_v = \prod_{(y',w)|(y,v)} k(y')_w$. $\longrightarrow$ Local ver.

$$\shortparallel$$

$$H^0_{dR}((X_y)_{K_v}/E_{y,v})$$

(4) Set $V_{y,v} = H^1_{dR}((X_y)_{K_v}/K_v) = H^1_{dR}((X_y)_{K_v}/E_{y,v})$

$$\circlearrowleft E_{y,v} \qquad \circlearrowleft E_{y,v}/K_v \quad (\Omega^1_{E_y/K} = 0).$$

$\forall (y',w)|(y,v)$,

$$V_{y',w} = H^1_{dR}((X_{y'})_{k(y')_w}/k(y')_w).$$

(5) All in all,

$$V_{y,v} = \bigoplus_{(y',w)|(y,v)} V_{y',w}.$$

$$\circlearrowleft \qquad\qquad \circlearrowleft$$

$$E_{y,v} \qquad\qquad E_{y',w} = \Gamma(y', \mathcal{O}_{Y',y'})_w.$$

§2 On period maps

Step 1 Setup Φ_v. ($y \in Y(K) \cap \Omega_v$).

Want: p-adic Hodge theory $\rightsquigarrow$ $\boxed{\begin{matrix}\text{cris rep'n } \rho_y|_{G_{K_v}} \\ \updownarrow \\ \text{filtered } \varphi\text{-mod}\end{matrix}}$ Key

via cohom comparisons.

- $V_{y,v}$ free $E_{y,v}$-mod, rk $= 2d$. w/ symp form ω, crys Frob φ_v.
- Observe $F^1 V_{y,v}$ = Lagrange submod, dim d.

1st piece in Hodge fil'n. i.e. $\omega|_{F^1V_{y,v} \times F^1V_{y,v}} = 0$.

Gauss-Manin $\rightsquigarrow$ $E_{y,v} \cong E_{y_0,v}$

$$(V_{y,v}, \omega, \varphi_v) \cong (V_{y_0,v}, \omega, \varphi_v).$$

$$\rightsquigarrow \quad F^1 V_{y,v} \longmapsto \text{Lag submod} \circlearrowleft E_{y_0,v}$$

$\rightsquigarrow$ refined period mapping

$$\mathcal{G}_v = \mathrm{Res}^{E_{y_0,v}}_{K_v} Gr(V_{y_0,v}, \omega)$$

$$\Phi_v : \Omega_v \longrightarrow \mathcal{H}_v = \mathrm{Res}^{E_{y_0,v}}_{K_v} LGr(V_{y_0,v}, \omega) \quad \text{flag var.}$$

$$y \longmapsto Fil^1 V_{y,v}. \qquad \circlearrowleft \varphi_{E_{y_0,v}}.$$

Explaination $X_{y_0} \to \mathrm{Spec}\, E_{y_0} \to \{y_0\}$.

$Fil^1 = E_{y_0}$-liner subspace & Lagrangian w.r.t. ω.

Rmk Can also setup Φ_w.

- GM-connection $\rightsquigarrow$ $\{(y', w) | (y,v)\} \cong \{(y_0', w_0) | (y_0, v)\}$

b/c $\prod_{(y',w)|(y,v)} K(y')_w = E_{y,v} \cong E_{y_0,v} = \prod_{(y_0',w_0)|(y_0,v)} K(y_0')_{w_0}$.

- $\mathcal{H}_v = \prod \mathcal{H}_{y_0',w_0}$, $\mathcal{H}_{y_0',w_0} = \mathrm{Res}_{K_v}^{k(y_0')_{w_0}} \mathrm{LGr}(V_{y_0',w_0}, \omega)$.
- $(V_{y',w}, \omega, \varphi_w) \cong (V_{y_0',w_0}, \omega, \varphi_{w_0})$

$$\rightsquigarrow \Phi_{y_0',w_0} : \Omega_v \xrightarrow{\Phi_v} \mathcal{H}_v \twoheadrightarrow \mathcal{H}_{y_0',w_0}$$
$$y \longmapsto \mathrm{Fil}^1 V_{y,v} \longmapsto \mathrm{Fil}^1 V_{y',w}. \qquad \text{(not necessary)}.$$

<u>Step 2</u> Local-global comparison.

- Target of period mapping : $\mathcal{H} := \mathrm{Res}_K^{E_{y_0}} \mathrm{LGr}(V_{y_0}, \omega)$.
- Choose $K \hookrightarrow \mathbb{C} \rightsquigarrow y_{0,\mathbb{C}} \in Y(\mathbb{C})$.

$$\boxed{E_{y_{0,\mathbb{C}}}} \cong \prod_{\substack{y_{\mathbb{C}}' \in Y'(\mathbb{C}) \\ y'|y_{0,\mathbb{C}}}} \mathbb{C}$$

$$\rightsquigarrow \Phi_{\mathbb{C}} : \tilde{Y}_{\mathbb{C}}^{an} \longrightarrow \mathcal{H}_{\mathbb{C}} \cong \mathrm{Res}_{\mathbb{C}}^{E_{y_{0,\mathbb{C}}}} \mathrm{LGr}(V_{y_{0,\mathbb{C}}}, \omega).$$

~~finite~~ unram cover $\to \downarrow$ $Y_{\mathbb{C}}^{an}$

$$= \prod_{y_{\mathbb{C}}'|y_{0,\mathbb{C}}} \mathrm{LGr}(V_{y_{\mathbb{C}}'}, \omega).$$

<u>Upshot</u> Lemma 3.2 : $\boxed{\overline{\mathrm{Im}\,\Phi_v} \text{ "=" } \overline{\mathrm{Im}\,\Phi_{\mathbb{C}}}}$.

i.e. $\exists\, Z \subseteq \mathcal{H}$ closed subvar s.t. $Z_v = \overline{\mathrm{Im}\,\Phi_v}$,
$Z_{\mathbb{C}} = \overline{\mathrm{Im}\,\Phi_{\mathbb{C}}}$.

- Full monodromy assumption

$$\Rightarrow \overline{\mathrm{Im}(\pi_1(Y_{\mathbb{C}}, y_{0,\mathbb{C}}) \to GL(V_{y_{0,\mathbb{C}}}))}^{Zar} \supseteq \prod_{y_{\mathbb{C}}'|y_{0,\mathbb{C}}} \mathrm{Sp}(V_{y_{\mathbb{C}}'}, \omega)$$

$$\overset{\shortparallel}{\Gamma}$$

$$\left(\begin{array}{l} \Gamma \curvearrowright V_{y_{\mathbb{C}}'} \text{ transitively} \\ \Rightarrow \Gamma \curvearrowright \mathcal{H}_{\mathbb{C}} = \mathrm{LGr}(V_{y_{0,\mathbb{C}}}, \omega) \text{ transitively} \\ \qquad\qquad H_{dR}^1(X_{y_0}(\mathbb{C})) = \prod_{y_{\mathbb{C}}'|y_{0,\mathbb{C}}} V_{y_{\mathbb{C}}'}. \end{array} \right)$$

$$\Rightarrow \underset{\overset{\shortparallel}{Z_{\mathbb{C}}}}{\overline{\mathrm{Im}\,\Phi_{\mathbb{C}}}} = \mathcal{H}_{\mathbb{C}} \Rightarrow Z = \mathcal{H}$$
$$\Downarrow$$
$$Z_v = \mathcal{H}_v$$

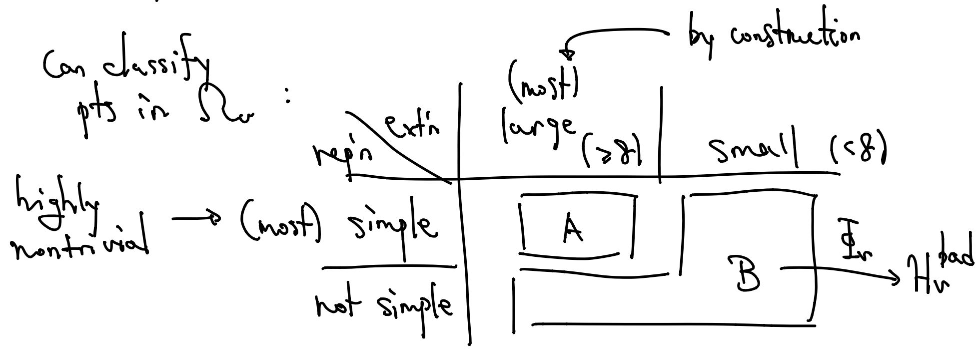
④ <u>Thus</u> $\Phi_v : \Omega_v \to \mathcal{H}_v$ has dense image $\mathrm{Im}\,\Phi_v \subseteq \mathcal{H}_v$.

§3 Main strategy

$\Phi_v : \Omega_v \to \mathcal{H}_v$. Fix $y \in Y(K)^* \cap \Omega_v$.

<u>Goal</u> $\dim_{K_v}\left(\overline{\underbrace{\mathrm{Cent}(\phi_v)}_{\text{centralizer of crys Frob } \phi_v} \cdot \Phi_v(y)}\right) < \dim_{K_v}\left(\overline{\mathrm{Im}\,\Phi_v}\right)$. $\quad (*)$

<u>Hopefully</u> Can classify pts in Ω_v:



- Faltings's lem $\Rightarrow$ A has $< \infty$ isom classes
- Lem 6.2 $\Rightarrow$ each isom class has $< \infty$ pts.

} p-adic Hodge theory

- Lem 6.1 $\Rightarrow$ $\# B < \infty$. (didn't use p-adic Hodge theory).

$\boxed{\text{Big issue}}$ Cannot conclude by Faltings's lemma everywhere b/c ρ_y not necessarily ss.

($\Rightarrow$ $y \mapsto [\rho_y|_{G_{K_v}}]$ infinite image) even if it has finite fibers

<u>Note</u> $(*) \Rightarrow \{y \in \Omega_v,$ attached w/ $\rho_y|_{G_{K_v}}\} \subseteq$ proper Zar closed subsch of Ω_v.

$$\{y \in \Omega_v, \text{ attached w/ } \rho_y|_{G_{K_v}}\} = \Phi_v^{-1}\big(\underbrace{\mathrm{Cent}(\phi_v)\cdot\Phi_v(y)}_{\cap\, \mathcal{H}_v}\big)$$

Ω_v curve $\Rightarrow \dim_{K_v}\{y \in \Omega_v \text{ w/ } \rho_y|_{G_{K_v}}\} = 0$.

$\Rightarrow \Phi_v^{-1}(\mathrm{Cent}(\phi_v)\cdot\Phi_v(y))$ finite.

$\Rightarrow y \mapsto [\rho_y|_{G_{K_v}}]$ has finite fibers.

pf idea of (*)

(1) "Im Φ_v is large" by comparison b/w $\Phi_v, \Phi_{\mathbb{C}}$.

$Z(\phi_v) \rightsquigarrow \Phi_v$ has Zar dense image.

(2) "$\overset{Z(\phi_v)}{\overset{\|}{\text{Cent}}}\ \phi_v$ is small". $\Leftarrow$ $\text{size}_v(\pi^{-1}(y_0)) < \frac{1}{d+1}$.

ϕ_v semilinear & most ext'ns $K(y')/K(y)$ are large

$\rightsquigarrow$ can bound $\dim_{\mathbb{Q}_p}(Z(\phi_v))$.

(3) Anyway, assume $E_{y_0,v} = K_w$ deg e ext'n of K_v.

$$\dim_{K_v}(\overline{\text{Im}\,\Phi_v}) = \dim_{K_v} \mathcal{H}_v \ , \quad \mathcal{H}_v = \text{Res}_{K_v}^{K_w} LGr.$$
$$= [K_w : K_v] \cdot \dim_{K_w} LGr$$
$$= e \cdot \frac{1}{2} \cdot d(d+1).$$

By contrast,

Lem 2.1 $\Rightarrow \dim_{K_v} Z(\varphi_w) = \dim_{K_w}(Z(\phi_w^e)) \leq (2d)^2$.

So $e \geq 8 \Rightarrow 8 \cdot \frac{1}{2} d(d+1) \geq (2d)^2 \Rightarrow$ (*).

§4 Gal rep'ns really do vary

Lem 6.2 Fix K_v'/K_v of deg ≥ 8. Fix ρ' of $G_{K_v'}$.

$$\Rightarrow \# \left\{ y \in \Omega_v \cap Y(K) : \begin{array}{l} \exists\,(y',w) \,|\, (y,v) \text{ w/ large ext'n and simple rep'n} \\ \text{s.t. } (K(y')_w, \rho_{y'}|_{G_{K(y')_w}}) \cong (K_v', \rho') . \end{array} \right\} < \infty.$$

Prof Known: • $\rho_{y'}|_{G_{K(y')_w}} \longleftrightarrow (V_{y',w}, \varphi_w, \text{Fil}^1 V_{y',w}) \longleftrightarrow \Phi_{y_0',w_0}(y)$

• Gauss-Manin $\Rightarrow (y',w)$ above $(y,v) \longleftrightarrow (y_0',w_0)$ above (y_0,v).

• Consider $h \in \mathcal{H}_{y_0',w_0} \longleftrightarrow \rho'$. can replace with $Z(\varphi_{w_0}^{[K_v:\mathbb{Q}_p]})$ b/c $Z(\varphi_{w_0}) \subseteq Z(\varphi_{w_0}^{[K_v:\mathbb{Q}_p]})$

$\rightsquigarrow$ To show $\#\{ y \in \Omega_v \cap Y(K) : \Phi_{y_0',w_0}(y) \in \boxed{Z(\varphi_{w_0})} \cdot h \} < \infty$.

$\| \ \{ f \in GL(V_{y_0',w_0}) : f \circ \varphi_{w_0} = \varphi_{w_0} \circ f \}$.

Issue: Now φ_{w_0} Frob-semilin

$\Rightarrow Z(\varphi_{w_0}) / \mathbb{Q}_p$ v.s. $\underbrace{\text{but NOT } / K_v.}_{\text{need a modification.}}$

(But $Z(\varphi_{w_0}^{[K_v:\mathbb{Q}_p]})$ is v.s. $/ K_v$.)

$K'_{w_0} = K(y_0')_{w_0}$. **Modification**: $\Phi_{y_0',w_0}(y) \in Z(\varphi_{w_0}^{[K_v:\mathbb{Q}_p]})\cdot h$ instead.

$$\begin{array}{l} r \,|\, \geq 8 \\ K_v \\ n \,| \\ \mathbb{Q}_p \end{array}$$

$$\begin{aligned}
\rightsquigarrow \dim_{K_v}(Z(\varphi_{w_0}^n)) &= \dim_{K'_{w_0}}(\underbrace{Z(\varphi_{w_0}^{nr})}_{2d \text{ dim'l}}) \\
&\leq \dim_{K'_{w_0}}(GL(V_{y_0',w_0})) \leq (2d)^2 < \tfrac{1}{2}\cdot 8d(d+1) \\
&\overset{r\geq 8}{\leq} r\cdot \dim_{K'_{w_0}}(\underbrace{LGr(V_{y_0',w_0},\omega)}_{=\frac{1}{2}d(d+1)}) \\
&\overset{\Phi_{w_0} \text{ dense image}}{=} \dim_{K_v}(\mathcal{H}_{y_0',w_0}).
\end{aligned}$$

$$\Rightarrow \underbrace{\{y \in \Omega_v \cap Y(K) : \Phi_{y_0',w_0}(y) \in Z(\varphi_{w_0}^n)\cdot h\}}_{0\text{-dim'l} \Rightarrow \text{finite.}} \subseteq \underbrace{\Omega_v \cap Y(K)}_{\text{curve}} \quad \text{proper Zariski closed}$$

curve

$\square$

§5 Generic Simplicity

Lem 6.1 $\#B < \infty$. (All but finitely many pts in $\Omega_v \cap Y(K)^*$ w/ large ext'n & simple Gal rep'n.).

Rmk (Seems) No p-adic Hodge theory involved.

Pf: $\underbrace{\text{Elementary inequalities}}_{\text{tricky}}$ & $\underbrace{\text{lin alg}}_{\text{routine argument}}$

possibly GPT can do (?)

Sketch: Step 1 Drop simplicity condition. But still require large ext'n.

$\rightsquigarrow$ Sublem $B \xrightarrow{\Phi_v} \mathcal{H}_v^{bad} := \left\{ F : \begin{array}{l} \exists\, \varphi\text{-stable proper subrep'n } W \subseteq \mathcal{H}_v \\ \text{s.t. } \dim(F \cap W) \geq \frac{1}{2}\dim W. \end{array} \right\}$

Alternatively $y \in \Omega_v \cap Y(k)^*$, bad pt.

$\Rightarrow \exists\, (y', w) \mid (y, v)$ s.t. $[k(y')_w : k_v] \geq 8$

and $\exists$ proper subspace $\underset{\circlearrowleft\,\varphi_w}{W_{y',w}} \subseteq V_{y',w}$, $\dim \mathrm{Fil}^h V_{y',w} \geq \frac{1}{2} \dim V_{y',w}$.

($\dim = \dim_{k(y')_w}$).

Rmk (Technically) v friendly $\Rightarrow$ possible HT wts of ρ_y (or ρ) are constrained

$\Rightarrow$ they are crs at $v \mid p$

and pure of some wt.

pf of sublem: By elementary inequality calculus.

Step 2 Fix $(y_0', w_0) \mid (y_0, v)$ s.t. $[k(y_0')_{w_0} : k_v] \geq 8$.

Sublem $\Rightarrow$ suffices to prove

$$\#\{ y \in \Omega_v \cap Y(k) : \Phi_{y_0', w_0}(y) \in \overline{H^{bad}_{y_0', w_0}} \} < \infty.$$

parametrizes Lagrangian subspaces

$F \subseteq V_{y_0', w_0}$ s.t. $\exists\, 0 \neq \underset{\circlearrowleft\,\varphi_{w_0}}{W} \subsetneq V_{y_0', w_0}$

w/ $\dim F \cap W \geq \frac{1}{2} \dim W$.

Step 3 $H^{bad}_{y_0', w_0} \subseteq$ proper subvar of H_{y_0', w_0}.