Kodaira-Parshin Construction

qbg

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We work over $\mathbb{C}$ analytically.

(0.1) . We first construct, for a genus ≥ 2 curve Y and a finite center-free group G, the Hurwitz space $Y' = \mathcal{H}_{G,1}(Y)$ that parametrizes G-covers of Y branched at one point. Together with this we have a relative curve Z/Y' and a morphism $Z \longrightarrow Y' \times Y$.

(0.2) . Take $G = \mathrm{Aff}_q = \mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times$ we get the *Kodaira-Parshin curve family* $Z_q \longrightarrow Y_q' \longrightarrow Y$.

(0.3) . Take "relative reduced Prym" we get the *Kodaira-Parshin family* $X_q \longrightarrow Y_q' \longrightarrow Y$.

1. The Hurwitz space

Theorem (1.1). *Let X be a path-connected, locally path-connected and semilocally simply-connected topological space. Then covering spaces of X is in bijection with subgroups of $\pi_1(X, x)$.*

Theorem (1.2) (Riemann existence theorem)**.** *Let Y be a compact Riemann surface, G be a finite group and $y \in Y$ be a point. Then G-covers of Y branched at y are in bijection with conjugacy classes of surjections $\pi_1(Y - y, *) \longrightarrow G$ nontrivial around y.*

(1.3) . Let $S(y)$ denote the set of G-covers of Y branched only at y and let $Y' = \coprod S(y)$. $S(y)$ is finite because $\pi_1(Y - y, *)$ is finitely generated.

(1.4) . Let $\Delta \subseteq Y$ be a small disk around y, and $y' \in \Delta$. $\pi_1(Y - \Delta, *) = \pi_1(Y - y, *) = \pi_1(Y - y', *)$. Because fundamental group classifies coverings, we see that every covering of $Y - \Delta$ is the restriction of a unique covering of $Y - y$ (resp. $Y - y'$). This gives a way to identify $S(y)$ and $S(y')$: a covering of $Y - y$ and a covering of $Y - y'$ are identified if their restriction gives the same covering of $Y - \Delta$.

(1.5) . Then we can give topology on $Y' = \coprod_y S(y)$ s.t. Y' is a compact Riemann surface and the natural map $Y' \longrightarrow Y$, $y' \in S(y) \longmapsto y$ is étale. Each $y' \in Y$ represents a cover of Y, so recover these curves we get a relative curve Z/Y' and a morphism $Z \longrightarrow Y' \times Y$.

(1.6) . Take $G = \mathrm{Aff}_q = \mathbb{F}_q^+ \rtimes \mathbb{F}_q^\times$ we get the *Kodaira-Parshin curve family* $Z_q \longrightarrow Y'_q \longrightarrow Y$.

2. The relative reduced Prym variety, $X_q \longrightarrow Y'_q \longrightarrow Y$

(2.1) . Given $f : C_1 \longrightarrow C_2$ we have $f^* : J_{C_2} \longrightarrow J_{C_1}$. This map has finite kernel and its cokernel is called the *Prym variety* associated to f.

(2.2) . Now back to our setting, let $f : C_1 \longrightarrow C_2$ be a Aff_q-cover and branched only at one point of C_2. Rather that take its Prym directly we use a "reduced" version. We shall take a subgroup $H = \mathbb{F}_q^\times \hookrightarrow G = \mathrm{Aff}_q$ and consider the corresponding intermediate cover $C_1 \longrightarrow C'_1 \longrightarrow C_2$. We are interested in the Prym $\mathrm{coker}(J_{C'_1} \longrightarrow J_{C_2})$ and we want to reformulate this, in a way that can be applied in relative situation.

(2.3) . $J_{C_2} \longrightarrow J_{C_1}$ has finite kernel, and the image is the identity component of $J_{C_1}^G$. Similarly $J_{C'_1} \longrightarrow J_{C_1}$ has finite kernel, and the image is the identity component of $J_{C_1}^H$. So

$$\begin{aligned}\mathrm{coker}(J_{C'_1} \longrightarrow J_{C_2}) &\sim \text{identity component of } J_{C_1}^H/\text{identity component of } J_{C_1}^G \\ &\sim \text{identity component of } J_{C_1}^H/J_{C_1}^G.\end{aligned}$$

(2.4) . The RHS can be computed via idempotent method. Let's go back to those good old days of representation theory of finite groups. Let $V/\mathbb{Q}$ be a vector space with G-action. Then $V^H = V^G \oplus V^{e_H - e_G = \mathrm{id}}$, where $e_H = \frac{1}{\#H}\sum h$ and $e_G = \frac{1}{\#G}\sum g$ in $\mathbb{Q}[G]$.

(2.5) . First note that v is fixed by H (resp. G) $\Longleftrightarrow$ $e_H v = v$ (resp. $e_G v = v$). Every H-fixed vector v can be written as $v = e_H v = e_G v + (e_H - e_G)v$. This proves $V^H = V^G + V^{e_H - e_G = \mathrm{id}}$. The intersection consists of $v = e_G v = (e_H - e_G)v$, $v = 2v$, $v = 0$. This proves $V^H = V^G \oplus V^{e_H - e_G = \mathrm{id}}$.

(2.6) . Now you can believe that the identity component of $J_{C_1}^H/J_{C_1}^G$ is isogenous to the identity component of $J_{C_1}[\#G(1 - (e_H - e_G))]$.

(2.7) . The districption above can be applied in relative situation. From the Kodaira-Parshin curve family $Z_q \longrightarrow Y'_q$ we construct

$$X_q = \text{relative identity component of } J_{Z_q/Y'_q}[\#G(1 - (e_H - e_G))].$$

Then $X_q \longrightarrow Y'_q$ is an abelian scheme whose fiber over $y' \in Y'_q$ is isogenous to the reduced Prym of the covering $Z_{y'} \longrightarrow Y$.