Monodromy of Kodaira-Parshin family

Recall K number field.

Y = proj sm geom conn curve of genus ≥ 2 $/K$.

q = prime ≥ 3

$\rightsquigarrow \mathrm{Aff}(q) = \mathbb{F}_q \rtimes \underbrace{\mathbb{F}_q^\times}_{\text{centralizer of } 0 \in \mathbb{F}_q} \curvearrowright \mathbb{F}_q$ via $(a,b)(x) = ax+b$

$\rightsquigarrow \mathrm{Aff}(q) \subseteq \mathfrak{S}_q$.

Step 1 Consider the moduli problem

$$\begin{array}{ccc} \mathrm{Sch}/K & \longrightarrow & \mathrm{Sets} \\ S & \longmapsto & \left\{ (y, \tilde{Z}_S) \;\middle|\; \begin{array}{l} \text{An } S\text{-pt } y \in Y(S) \text{ and } \mathrm{Aff}(q)\text{-Galois cover} \\ \tilde{Z}_S \to Y - \{y\} \text{ s.t. monodromy at } y \text{ is nontrivial} \end{array} \right\} \end{array}$$

This is rep'd by Y_q' with universal obj

- a pt on Y: $Y_q' \xrightarrow{y} Y$ $/K$ (fin étale)
- $\mathrm{Aff}(q)$-cover

$$\begin{tikzcd} \tilde{Z}_q \arrow[d, "\text{proj sm family of cover}"'] \arrow[r, "\text{Aff(q)-cover, ramified along } \Gamma_y"] & Y \times Y_q' \supseteq \Gamma_y \arrow[dl] \\ Y_q' & \end{tikzcd}$$

(G-cover $Z \xrightarrow{f} Y \ni y$ at y : $Z \setminus f^{-1}(y) \to Y \setminus \{y\}$ fin étale G-torsor.)

Step 2 $\mathrm{Aff}(q) \curvearrowright \mathbb{F}_q \rightsquigarrow \mathrm{Aff}(q) \curvearrowright \mathbb{Q}[\mathbb{F}_q] = \mathbb{Q} \oplus \mathbb{Q}[\mathbb{F}_q]^\circ$

Claim The idempotent for the rep'n $\mathbb{Q}[\mathbb{F}_q]^\circ$ is

$$e := \frac{1}{\#\mathbb{F}_q^\times} \sum_{h \in \mathbb{F}_q^\times} h - \frac{1}{\#\mathrm{Aff}(q)} \sum_{g \in \mathrm{Aff}(q)} g$$

$\rightsquigarrow$ Put $e'' := \#\mathrm{Aff}(q) \cdot (1-e)$.

$\mathbb{Q}[\mathbb{F}_q]$

$= \mathbb{Q}[\mathrm{Aff}(\mathbb{F}_q)/\mathbb{F}_q^\times]$

$= \mathbb{Q}[\mathrm{Aff}(\mathbb{F}_q)]$.

Consider $X_q := \mathrm{Coker}(\mathrm{Pic}^0_{Y\times Y_q/Y_q} \longrightarrow \mathrm{Pic}^0_{\tilde{Z}_q/Y_q})[e']^\circ$

$\overset{\text{isog}}{\approx} \mathrm{Coker}(\mathrm{Pic}^0_{Y\times Y_q'/Y_q'} \longrightarrow \mathrm{Pic}^0_{Z_q/Y_q'}) = \mathrm{Prym}_{Z_q \text{ rel } Y\times Y_q'}$.

where $Z_q = \underset{\substack{\circlearrowleft \\ \mathrm{Aff}(q)}}{\tilde{Z}_q} \times_{\mathrm{Aff}(q)} \mathbb{F}_q \xrightarrow{q\text{-cover}} Y \times Y_q'$ ram @ Γ_y.

$\rightsquigarrow$ Kodaira–Parshin family:

$$\begin{array}{ccl} X_q & \leftarrow & \text{abelian variety.} \\ \downarrow & & \pi_q^{-1}(y) = \{y_1, \dots, y_N\} \\ & & \qquad\qquad \downarrow \\ Y_q & \xrightarrow{\pi_q} & Y \ni y \end{array}$$

$\mathrm{Cor}_i : \pi_1(Y - \{y\}, y_0) \to \mathrm{Aff}(q)$.

$\Updownarrow$

$\mathrm{Aff}(q)$-cover $\tilde{Y}_i \to Y$ of Y, ramified at y (up to isom).

Key point Fix $y \in Y(\mathbb{C})$. For each $\tilde{Y}_i$, define $Y_i := \tilde{Y}_i \times_{\mathrm{Aff}(q)} \mathbb{F}_q \xrightarrow{q\text{-cover}} Y$.

$\Rightarrow X_{q, y_i} \overset{\text{isog}}{\approx} \mathrm{Prym}_{Y_i \text{ rel } Y} = \mathrm{Coker}(\mathrm{Pic}^0_Y \to \mathrm{Pic}^0_{Y_i})$

$X_{q,y} \overset{\text{isog}}{\approx} \prod_i \mathrm{Prym}_{Y_i \text{ rel } Y}$.

Main thm $\pi_1(Y, y) \longrightarrow \prod_i \mathrm{Sp}(\underbrace{H_1^{\mathrm{Pr}}(Y_i \text{ rel } Y; \mathbb{Q})})$ has Zariski dense image

$H_1(Y_i, \mathbb{Q}) = H_1(Y, \mathbb{Q}) \oplus H_1^{\mathrm{Pr}}(Y_i \text{ rel } Y; \mathbb{Q})$

orthogonal decomp. w.r.t. intersection pairing.

Subtle point $\mathrm{Aff}(q) \subseteq G_q = \mathrm{Permation}(0, \dots, q-1)$

$N_{G_q}(\mathrm{Aff}\,q) = \mathrm{Aff}(q)$ (?)

So $\{\pi_1(Y - \{y\}, y_0) \twoheadrightarrow \mathrm{Aff}(q)\} / \mathrm{Aff}(q)\text{-Conj}$

$= \{\pi_1(Y - \{y\}, y_0) \circlearrowright \mathbb{F}_q$ whose action factors surjectively through $\mathrm{Aff}(q)\}/\sim$

$\leftrightarrow$ q-cover of $Y - \{y\}$ with monodromy gp $\mathrm{Aff}(q)$.

via $\varphi \xmapsto{\text{Conj}} (\pi_1 \to \mathrm{Aff}(q) \circlearrowright \mathbb{F}_q)$.

Goursat's lemma Suppose G is an alg subgp of $Sp(V)^N$ satisfying

- for $1 \le i \le N$, the proj $\pi_i : G \longrightarrow Sp(V)$ surj
- for $1 \le i < j \le N$, $\exists\, g \in G$ s.t. $\mathrm{rank}(\pi_i(g) - 1) \neq \mathrm{rank}(\pi_j(g) - 1)$.

Then $G = Sp(V)^N$.

Ribet's paper (original lemma)

If $A \subseteq B_1 \times B_2$ is a subgp s.t. $\pi_i : A \to B_i$ surj.

Let $N_2 = \ker(p_1 : A \to B_1) \subseteq \{1\} \times B_2$, $N_1 = \ker(p_2 : A \to B_2) \subseteq B_1 \times \{1\}$.

Then $N_i \trianglelefteq B_i$ is normal, and $\mathrm{Im}(A \to B_1/N_1 \times B_2/N_2)$ is a graph of an isom $B_1/N_1 \simeq B_2/N_2$.

Related tools If $g \in Sp(V)$ is unipotent s.t. $\mathrm{rank}(g - \mathrm{id}) = 1$.

then $g = T_v^r : x \mapsto x + r \cdot \langle v, x \rangle v$ for some $v \in V$, $r \in K$.

"transvection"

Basic tools to prove "large image":

Mapping class group + Dehn twist

(view Y as a topological real surface.)

Def'n For a top. real surface Y (allowing boundary & puncture),

define the mapping class gp of Y to be

$$MCG(Y) := \{\text{diffeomorphism of } Y\} / \text{isotropy}.$$

$$\rightsquigarrow MCG(Y) \longrightarrow \mathrm{Out}(\pi_1(Y, y_0)).$$

$$\{\mathrm{Cor} : \pi_1(Y, y_0) \to \mathrm{Aff}(g)\} / \mathrm{Aff}(g)$$

$\mathrm{Im}(\pi_1(Y, y_0))$ trivially.

For (Z,π) an $\mathrm{Aff}(q)$-cover of Y,

write $\mathrm{MCG}(Y)_Z$ for the stabilizer of this action.

- $\forall\, \eta \in \mathrm{MCG}(Y)_Z$ lifts uniquely to Z, (b/c $Z_{G_q}(\mathrm{Aff}(q)) = \{1\}$.)

i.e.
$$\begin{tikzcd} Z \arrow[r, dashed, "\eta"] \arrow[d, "\pi"'] & Z \arrow[d, "\pi"] \\ Y \arrow[r, "\eta"] & Y \end{tikzcd}$$

i.e. $\exists\ \mathrm{MCG}(Y)_Z \longrightarrow \mathrm{MCG}(Z) \curvearrowright H_1(Z,\mathbb{Q})$.
$$\eta \longmapsto \tilde{\eta}.$$

$\rightsquigarrow$ monodromy map $\mathrm{Mon}: \mathrm{MCG}(Y)_Z \longrightarrow \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z,Y))$.

<u>Back to our situation</u>:

Y = proj Riemann surface $\ni y$.

$\exists$ finitely many $Z_1, \dots, Z_n \xrightarrow{q\text{-cover}} Y$, $\mathrm{Aff}(q)$-cover, ram at y.

Put $\mathrm{MCG}(Y\setminus\{y\})_0 := \bigcap_i \mathrm{MCG}(Y\setminus\{y\})_{Z_i\setminus\{\text{fiber of } y\}}$.

$\cap$ fin index

$\mathrm{MCG}(Y\setminus\{y\})$

<u>Fact</u> $\exists$ exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(Y,y) \arrow[r] & \mathrm{MCG}(Y-\{y\}) \arrow[r] & \mathrm{MCG}(Y) \arrow[r] & 1 \\ & \pi_1(Y,y)_0 \arrow[r] \arrow[u, hook] & \mathrm{MCG}(Y-\{y\})_0 \arrow[r, "\mathrm{Mon}"] \arrow[u, hook, "\text{fin index}"'] & \prod \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i,Y)) & \end{tikzcd}$$

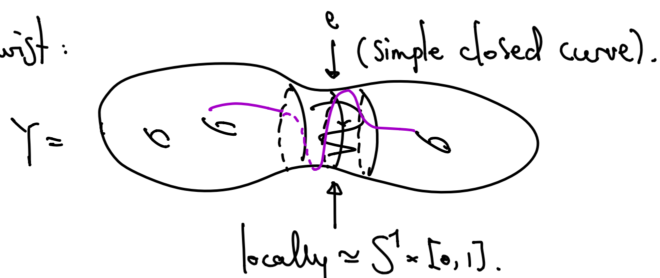
(Hard) $\mathrm{Mon}(\mathrm{MCG}(Y-\{y\})_0)$ has Zariski dense image in $\prod \mathrm{Sp}$.

(Easy) $\forall i$, $\pi_1(Y,y)_0 \xrightarrow{\mathrm{Mon}} \prod \mathrm{Sp} \xrightarrow{\mathrm{pr}_i} \mathrm{Sp}$

does not lie in the center.

$\Longrightarrow$ <u>Main thm</u> b/c $\pi_1(Y,y)_0$ is a normal subgp of $\mathrm{MCG}(Y-\{y\})_0$.

Dehn twist :



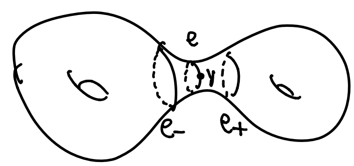
$D_e : Y \longrightarrow Y$ identity away from $S^1 \times [0,1]$ on $S^1 \times [0,1]$,

given by $(\theta, t) \longmapsto (\theta + 2\pi t, t)$.

D_e acts on $H_1(Y)$ by transvection $x \longmapsto x + r \cdot \langle x, e \rangle e$.

<u>Rmk</u> The map $\pi_1(Y, y) \longrightarrow MCG(Y - \{y\})$ can be interpreted as

$$e \longmapsto D_{e_+} \circ D_{e_-}^{-1}$$



<u>Discussion</u> How to lift $D_e^{n_e}$ to $MCG(Z)$?

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ MCG(Y-\{y\})_Z \end{array}$$

$$\pi_1(Y, y) \longrightarrow Aff(\varphi) \longrightarrow \mathfrak{S}(\mathbb{F}_q)$$
$$e \longmapsto \sigma_e.$$

if σ_e has permutation type $(k_1, \cdots, k_r)$

$\Rightarrow \pi^{-1}(e) = e_1 \sqcup e_2 \sqcup \cdots \sqcup e_r$, $e_i \xrightarrow{k_i} e$.

If n_e is divisible by all k_i

$\Rightarrow D_e^{n_e}$ can be lifted to $\prod D_{e_i}^{n_e/k_i} \in MCG(Z)$.

<u>Observation</u> $D_e^{n_e}$ = product of transvection along $e_1, \cdots, e_r$

D_e is a transvection,

$\operatorname{rank}(\operatorname{Mon}(D_e^{n_e}) - id) = r - 1$.