Multivariable (φ,Γ)-modules and Modular Representations of Galois and GL_2.

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§1 Introduction

$F/\mathbb{Q}$ totally real, D/F = quaternion alg ramified at all ∞ places except $\mathfrak{s}$.

$X_{U/F}$ = Shimura curve of level U associated to D

U = c.o. subgrp of $(D\otimes_F \mathbb{A}_F^{\infty})^\times$.

p = prime number, $v|p$ = place of F where $D_v = M_2(F_v)$.

General aim

$$\varinjlim_{U_v} H^1_{et}(X_{U^v U_v \times F}(\bar{F}), \bar{\mathbb{F}}_p)$$

where $U^v :=$ fixed compact open subgroup of $(D\otimes_F \mathbb{A}^{\infty,v})^\times$

$U_v :=$ compact open subgroup of $GL_2(F_v) = D_v^\times$.

$\bar{r} : Gal(\bar{F}/F) \longrightarrow GL_2(\bar{\mathbb{F}}_p)$ irred Gal rep'n

$\rightsquigarrow \pi_v(\bar{r}) := \varinjlim_{U_v} Hom_{Gal(\bar{F}/F)}(\bar{r}, H^1_{et}(X_{U^v U_v \times}(\bar{F}), \bar{\mathbb{F}}_p))$

$\circlearrowleft GL_2(F_v)$

For simplicity, assume:

v = unique p-adic place of F

U^v is as big as possible s.t. $\pi_v(\bar{r}) \neq 0$.

Some results
- $F=\mathbb{Q}$, $\pi_v(\bar{r})$ is known (Breuil, Colmez, Emerton, Berger, Kisin, …)
- $\pi_v(\bar{r})$ determines $\bar{r}_v$ (Scholze).

- F_v is unramified + $\bar{r}_v$ irred. (+ generic condition)
  $\Rightarrow$ $\pi_v(\bar{r})$ is irred.

<u>Rmk</u> if $F_v = \mathbb{Q}_p$, $\pi_v(\bar{r})$ "should" only depend on $\bar{r}_v$.
$F_v \neq \mathbb{Q}_p$, we do not know this.

Suppose $F_v = \mathbb{Q}_p$. A key ingredient associated to $\pi_v(\bar{r})$ is a rank 2 (φ, Γ)-module (Colmez).
When F_v is unramified, will associate a certain multivariable (φ,Γ)-module + state a precise conjecture.

§2 The main conjecture

Fix K = finite unram ext'n of $\mathbb{Q}_p$ with deg f, $q := p^f$.
$\mathbb{F}$ = finite ext'n of $\mathbb{F}_p$ (= coefficient field).

- <u>The ring A</u>

$\mathbb{F}[\![\mathcal{O}_K]\!] = \mathbb{F}[\![Y_\sigma, \sigma : \mathbb{F}_q \hookrightarrow \mathbb{F}]\!]$,
where $Y_\sigma = \sum_{\lambda \in \mathbb{F}_q^\times} \sigma(\lambda)^{-1} [\lambda]$. $[\cdot]$ = Teichmüller lifting.

Multiplying by p $\rightsquigarrow$ $\mathbb{F}$-linear Frob map φ.
Multiplying by $\mathcal{O}_K^\times$ $\rightsquigarrow$ $\mathbb{F}$-linear action of $\mathcal{O}_K^\times$.

When $f=1$:

$\mathbb{F}[\![\mathbb{Z}_p]\!][\frac{1}{Y}] = \mathbb{F}(\!(Y)\!)$

$\mathbb{F}[\![\mathcal{O}_K]\!][\frac{1}{Y_\sigma}]_\sigma$ but $\mathcal{O}_K^\times$ doesn't act!

To remedy this:

$A := \left(\mathbb{F}[\![\mathcal{O}_K]\!][\frac{1}{Y_\sigma} : \sigma]\right)^{\wedge}$ ← completion for the "(Y_σ, σ)-adic topology".

$\mathcal{O}_K^\times, \varphi$ ↻ = affinoid alg over $\mathbb{F}(\!(Y_\sigma)\!)$.

• $D_A(\pi)$

π = sm adm rep'n of $GL_2(K)/\mathbb{F}$,

$\pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ = module over $\mathbb{F}[\![I_1]\!]$ + $\mathfrak{m}_{I_1}$-adic topology

(I_1: pro-p Iwahori.)

$D_A(\pi) := \left(A \otimes_{\mathbb{F}[\![\begin{pmatrix}1 & \mathcal{O}_K\\ 0 & 1\end{pmatrix}]\!]} \pi^\vee\right)^{\wedge}$ ($\wedge$: completion for the tensor product topology.)

= A-module

+ semi-linear action of $\mathcal{O}_K^\times$ via $\begin{pmatrix}\mathcal{O}_K^\times & 0\\ 0 & 1\end{pmatrix} \curvearrowright \pi^\vee$.

+ $\tilde{\psi} : D_A(\pi) \longrightarrow A \otimes_{\varphi, A} D_A(\pi)$

induced by $f \mapsto f \circ \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$, $f \in \pi^\vee$.

<u>Conjecture (Part 1)</u> If $\pi = \pi_v(\bar{r})$ ($\Rightarrow \tilde{\psi}$ is bijective)

and $D_A(\pi_v(\bar{r}))$ is free of rank 2^f.

$$\varphi : D_A(\pi_v(\bar{r})) \hookrightarrow A \otimes_{\varphi, A} D_A(\pi_v(\bar{r})) \xrightarrow{\tilde{\psi}^{-1}} D_A(\pi_v(\bar{r}))$$

$\rightsquigarrow D_A(\pi_v(\bar{r})) + \varphi + \mathcal{O}_K^\times$ = étale $(\varphi, \mathcal{O}_K^\times)$-module.

• <u>The module $D_A^{\otimes}(\bar{\rho})$</u>

Take $\bar{\rho} : \mathrm{Gal}(\bar{K}/K) \longrightarrow GL_2(\mathbb{F})$

<u>Natural idea</u> use Fontaine's Lubin-Tate $(\varphi, \mathcal{O}_K^\times)$-module $D_{LT}(\bar{\rho})$.

$= \mathbb{F}(\!(T)\!)$-v.s. + φ_q-action + $\mathcal{O}_K^\times$-action.

$\mathcal{O}_K^\times$ ↺ Lubin-Tate action

φ_q via $\varphi_q(T) = T^q$.

<u>Problem</u> seems that to compare $\mathbb{F}[\![T]\!]$ & $\mathbb{F}[\![\mathcal{O}_K]\!]$

b/c the $\mathcal{O}_K^\times$ actions look different.

Salvation comes from:

Theorem (Fargues, Fargues–Fontaine)

Consider $\underbrace{\mathbb{F}[\![T^{p^{-\infty}}]\!] \hat{\otimes}_{\mathbb{F}} \cdots \hat{\otimes}_{\mathbb{F}} \mathbb{F}[\![T^{p^{-\infty}}]\!]}_{f \text{ copies}} = \mathbb{F}[\![T_0^{p^{-\infty}}, \dots, T_{p-1}^{p^{-\infty}}]\!]$.

$\circlearrowleft$ $(K^\times)^f \rtimes S_f$ (p acts via φ_f)

Then there is a natural isomorphism

$$m : \mathbb{F}[\![Y_\sigma^{p^{-\infty}} : \sigma]\!] \xrightarrow{\sim} \mathbb{F}[\![T_0^{p^{-\infty}}, \dots, T_{f-1}^{p^{-\infty}}]\!]^{\Delta \rtimes S_f}$$

where $\Delta := \{ (h_i) \in (K^\times)^f : \prod h_i = 1 \}$

compatible with $K^\times$-actions on both sides.

LHS: p acts via φ

RHS: residual action, $K^\times = (K^\times)^f \rtimes S_f / (\Delta \rtimes S_f)$.

$\rightsquigarrow$ Define $A_\infty := \left(\mathbb{F}[\![Y_\sigma^{p^{-\infty}} : \sigma]\!] \left[\frac{1}{Y_\sigma} : \sigma \right] \right)^\wedge$

$= $ an affinoid perfectoid $\mathbb{F}$-algebra.

$$\begin{array}{ccccc} \mathrm{Spa}(\mathbb{F}[\![T_i^{p^{-\infty}}, i]\!]) & \underset{\text{open}}{\hookleftarrow} & \mathrm{Spa}(\mathbb{F}(\!(T^{p^{-\infty}})\!), \mathbb{F}[\![T^{p^{-\infty}}]\!])^{\times_{\mathbb{F}} f} & \underset{\text{open}}{\hookleftarrow} & m^{-1}(\mathrm{Spa}(A_\infty, A_\infty^\circ)) \\ \downarrow m & & & & \downarrow m \quad \Delta \rtimes S_f\text{-action} \\ \mathrm{Spa}(\mathbb{F}[\![Y_\sigma^{p^{-\infty}}, \sigma]\!]) & & \xleftarrow{\text{open}} & & \mathrm{Spa}(A_\infty, A_\infty^\circ) \end{array}$$

A_∞°: power bounded elements

Let's go:

$\bar{\rho} \rightsquigarrow D_{LT}(\bar{\rho}) \rightsquigarrow \mathbb{F}(\!(T^{p^{-\infty}})\!) \otimes_{\mathbb{F}(\!(T)\!)} D_{LT}(\bar{\rho})$.

$\overset{\text{self-}\otimes}{\rightsquigarrow}$ locally free sheaf on $\mathrm{Spa}(\mathbb{F}(\!(T^{p^{-\infty}})\!), \mathbb{F}[\![T^{p^{-\infty}}]\!])^{\times f}$

$\rightsquigarrow$ $(K^\times)^f \rtimes S_f$-equivariant locally free sheaf on $m^{-1}(\mathrm{Spa}(A_\infty, A_\infty^\circ))$

$\overset{\text{descent}}{\rightsquigarrow}$ $K^\times$-equiv. locally free sheaf on $\mathrm{Spa}(A_\infty, A_\infty^\circ)$

(Scholze–Weinstein)

$\rightsquigarrow$ free A_∞-module $D^{\otimes}_{A_\infty}(\bar\rho)$ + $K^\times$-action

where $D^{\otimes}_{A}(\bar\rho)$ = étale $(\varphi, \mathcal{O}_K^\times)$-module free of rk $(\dim\bar\rho)^f$

(Kedlaya-Liu & Quillen-Suslin for A_∞)

Frob descent $\rightsquigarrow$ $D^{\otimes}_{A_\infty}(\bar\rho) = A_\infty \otimes_A D^{\otimes}_A(\bar\rho)$ where $D^{\otimes}_A(\bar\rho)$ = étale $(\varphi, \mathcal{O}_K^\times)$-module. free of rk $(\dim\bar\rho)^f$.

$\rightsquigarrow$ $\bar\rho \longmapsto D^{\otimes}_A(\bar\rho)$.

Theorem 1

① $\mathbb{F}((y)) \otimes_{A}^{(\mathrm{tr})} D^{\otimes}_A(\bar\rho) \simeq (\varphi, \Gamma)$-module of the lemma induction from K to $\mathbb{Q}_p$ of $\bar\rho$.

$\mathrm{tr}: \mathcal{O}_K \to \mathbb{Z}_p$

② If $\bar\rho$ is semisimple, $D^{\otimes}_A(\bar\rho)$ can be made completely explicit.

Conjecture (Part 2) $D_A(\pi_v(\bar r)) \simeq D^{\otimes}_A(\bar r_v(1))$ ← Tate twist

§3 The main results

Theorem 2 Assume · standard assumptions as the global setting (e.g. $\bar r|_{\mathrm{Gal}(\bar F/F[\cdots])}$ irred.).

· $\bar r_v$ is semi-simple and generic (in the following sense):

either $\bar r_v|_{\text{inertia}} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & 1 \end{pmatrix} \otimes (\text{twist})$

or $\bar r_v|_{\text{inertia}} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & \omega_{2f}^{q\cdot\sum_{i=0}^{f-1}(r_i+1)p^i} \end{pmatrix} \otimes (\text{twist})$

$p \geq 4f+1$ & $p \geq 29$

$\begin{cases} \max(12, 2f-1) \leq r_i \leq p - \max(15, 2f+2) & \text{if } i>0 \text{ or } \bar r_v \text{ irred.} \\ \max(13, 2f) \leq r_0 \leq p - \max(14, 2f+1) & \text{if } \bar r_v \text{ irred.} \end{cases}$

Then the conjecture is true (both parts).

<u>Sketch of proof</u> Long calculation:

⓪ We knew that $D_A(\pi_v(\bar{r}))$ is free of rank 2^f.

① $\mathrm{Hom}_A(D_A(\pi_v(\bar{r})), A) \hookrightarrow \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(D_A(\pi_v(\bar{r})), \mathbb{F})$ ← discrete.

induced by $\mu: A \to \mathbb{F}$ uniquely (determined by $\mu \circ \varphi \in \mathbb{F}^\times \mu$.

$= \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(\pi_v(\bar{r})^\vee[\frac{1}{Y_\sigma} : \sigma], \mathbb{F})$.

② Def'n of $D_A(\pi_v(\bar{r}))$ + weight cycling

⇝ can define 2^f natural elements $(n_\tau)_\tau$.

in $\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(\pi_v(\bar{r})^\vee[\frac{1}{Y_\sigma}], \mathbb{F})$. Serre weights of $\bar{\rho}$ (↑ τ)

③ (Subtle part) We prove that

$$\mathrm{Hom}_A(D_A(\pi_v(\bar{r})), A) = \bigoplus_\tau A \cdot n_\tau.$$

The genericity assumption applied.

④ ③ ⇝ explicit description of $D_A(\pi_v(\bar{r}))$

Thm 1 ② ⇝ explicit description of $D_A^{\otimes}(\bar{r}_v(1))$

Comparison ⇒ exactly the same. □

<u>Rmk</u> There should exist a more conceptual proof (without these general assumptions).

<u>Rmk</u> $\bar{\rho} \mapsto D_A^{\otimes}(\bar{\rho})$ is <u>not</u> fully faithful. (even in dim 2).

But if you fix $\det(\bar{\rho})(p)$, it's ok.