Lectures on Mod p Langlands Program for $GL_2(L)$ (1/4)

Yongquan Hu

<u>References</u>: Breuil-Herzig-Hu-Morra-Schraen (1), (2)
(1) Gelfand-Kirillov dim
(2) (φ,Γ)-mod
Hu-Wang

$\bar{\rho}$ semi-simple.

§ Introduction

$L/\mathbb{Q}_p$ finite ext'n.

$$\bar{\rho}: G_L \longrightarrow GL_2(\bar{\mathbb{F}}_p) \overset{?}{\longleftrightarrow} \pi(\bar{\rho}): \text{sm adm rep'n of } GL_2(L).$$
$$\bar{\mathbb{F}}_p \supseteq \mathbb{F}$$

Known for $GL_2(\mathbb{Q}_p)$: Breuil, Colmez, Emerton

$\bar{\rho}$: irred. $\longleftrightarrow$ $\pi(\bar{\rho})$ supersingular (Breuil)

$\bar{\rho}$: reducible $\longleftrightarrow$ $\pi(\bar{\rho})$: $PS_1 \longrightarrow PS_2$.

When $L \neq \mathbb{Q}_p$, no classification for <u>s.s.</u>

- Breuil-Paskunas (2007) (infinite) family of s.s. rep.
- Hu, Schraen, Wu: s.s. are not of finite rep'n

$$0 \to \mathrm{Ker} \longrightarrow \text{c-}\mathrm{Ind}_{GL_2(\mathcal{O}_L)}^{GL_2(L)} \sigma \longrightarrow \pi \to 0$$

(not finite type) $\qquad GL_2(L)$-rep'n.

- Le: $\exists$ non-adm smooth irred. s.s. rep.

<u>Candidate</u>: for $\pi(\bar{\rho})$, F tot real field.

D/F quaternion alg split above p;

at ∞: either non-split or split at only one place above ∞

$\bar{r} : G_F \to GL_2(\mathbb{F})$ cont. odd,

$\bar{r}|_{G_{F_v}} \simeq \bar{\rho}$, $U^v \subseteq (D \otimes \mathbb{A}_F^{\infty,v})^\times$ compact open

($F_v \simeq L$, $v|p$)

$\rightsquigarrow$ $\pi_v^D(\bar{r}) := \{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U^v \to \mathbb{F} \text{ cont} \} [m_{\bar{r}}]$.

$\circlearrowleft$ $GL_2(L)$? $\|$

eigenspace for $m_{\bar{r}}$ $\pi(\bar{\rho})^{\oplus d}$ (assume $d=1$)

(max'l ideal of Hecke alg).

<u>Goal</u> · property of $\pi(\bar{\rho})$? (finite length).

· "locality" of $\pi(\bar{\rho})$?

<u>Conjectural properties of $\pi(\bar{\rho})$</u>?

(1) As $K := GL_2(\mathcal{O}_L)$ - rep'n $\twoheadrightarrow GL_2(\mathbb{F}_q)$, $\mathcal{O}_L/(p) = \mathbb{F}_q$.

· Buzzard-Diamond-Jarvis (weight part for Serre conj).

$\mathrm{Soc}_K \pi(\bar{\rho}) = \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$. $W(\bar{\rho}) :=$ set of irred Γ-rep.

· [BP] Let $K_1 = \ker(K \twoheadrightarrow \Gamma)$,

unramified case:

$\overset{?}{\Rightarrow}$ $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ fin. dim Γ-rep'n.

proved by Gee, Emerton-Gee-Savitt, Le

(ver. with multiplicities) all n_σ's are equal),

(by using a patching functor)

(2) As $GL_2(L)$-rep'n:

[BP] [· $\pi(\bar{\rho})$ is generated by $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$ (i.e. finitely generated).

· $\pi(\bar{\rho})$ has finite length: $\begin{cases} 1 & \text{if } \bar{\rho} \text{ irred, ok!} \\ f+1 & \text{if } \bar{\rho} \text{ is reducible (generic)} \end{cases}$

$f=2$, ok ; $\pi_0 - \underbrace{\pi_1 - \cdots}_{\text{S.S.}} - \pi_f$ (?)

π_0: PS, π_f: PS

(Emerton's Conjecture (?):

f.g. + adm. $\Rightarrow$ fin length.)

- $\pi(\bar{\rho})$ has Gelfand-Kirillov dim f.

<u>Recall</u> $K_n = 1 + p^n M_2(\mathcal{O}_L)$, $\dim_{\mathbb{F}} \pi(\bar{\rho})^{K_n}$.

$$\exists\ 0 \le \underset{\substack{\uparrow \\ \text{integer}}}{c} \le \underset{\substack{\| \\ 4f}}{\dim K}, \quad a \ge b > 0$$

$$\rightsquigarrow\ bp^{nc} + O(p^{n(c-1)}) \le \dim_{\mathbb{F}} \pi(\bar{\rho})^{K_n} \le ap^{nc} + O(p^{n(c-1)})$$

$$c := GK\text{-}(\pi(\bar{\rho})).$$

<u>E.g.</u> - PS $\dim (\mathrm{Ind}_{B(L)}^{GL_2(L)} \chi)^{K_n} = p^{(n-1)f}(p^f+1) = p^{nf} + p^{(n-1)f}$. $GK = f$.

- $GL_2(\mathbb{Q}_p)$ (Morra):

$$\dim \pi^{K_n} = (p+1)(2p^{n-1}+1) + \begin{cases} p-3 \\ p-2 \end{cases} \longleftarrow r \in \{0, p-1\}, \quad \pi \text{ s.s.}$$

$$\Rightarrow\ GK(\pi) = 1.$$

- GK-$\dim = 0$ iff $\dim_{\mathbb{F}} \pi < \infty$.

<u>Rmk</u> $\mathbb{F}[[K_1]]$ local ring. $M = \pi(\bar{\rho})^\vee$ is f.g. $\mathbb{F}[[K_1]]$-mod

$$\pi(\rho)^{K_1} \longleftrightarrow M/\mathfrak{m}_{\mathbb{F}[[K_1]]} \subseteq M/\mathfrak{m}^2_{\mathbb{F}[[K_1]]}$$

<u>Rmk</u> The importance of GK-dim: ($\Rightarrow M_\infty$ is flat over K_∞).

patching $\rightsquigarrow M_\infty/$ over $R_\infty = R_{\bar{\rho}}[[x_1, \cdots, x_g]]$ univ. def ring.

satisfies $M_\infty/\mathfrak{m}_{R_\infty} \simeq \pi(\bar{\rho})^\vee$.

$\forall\ x : R_\infty \to \overline{\mathbb{Q}}_p$

$$\underset{?}{0 \ne} (\underbrace{M_\infty \otimes_{R_\infty, x} \mathcal{O}_E}_{p\text{-torsion}})^d [\tfrac{1}{p}] \xleftrightarrow{p\text{-adic}} \rho_x^{\text{univ}}.$$

$(-)^d = \mathrm{Hom}^{\text{cont}}_{\mathcal{O}_E}(-, \mathcal{O}_E)$ duality.

§ Serre Weight

$$\begin{cases} \text{(i) Serre weight: } \pi(\bar\rho)^{K_1}, \cdots \\ \text{(ii) GK-}\dim(\pi(\bar\rho)). \end{cases}$$

$\bar\rho \rightsquigarrow$ define the modular weight

$$W^{?}(\bar\rho) = \{\sigma \text{ irred } \Gamma\text{-rep or } K\text{-rep} \mid \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0\}$$

(all $\mathrm{soc}_K(\pi(\bar\rho))$.)

[BDJ] construct $W^{\mathrm{expl}}(\bar\rho)$.

Notation: irred σ has the form $\quad (\Gamma = GL_2(\mathbb{F}_q))$

$$(r_0, \cdots, r_{f-1}) \otimes \det{}^a := \mathrm{Sym}^{r_0}\mathbb{F}^2 \otimes (\mathrm{Sym}^{r_1})^{\mathrm{Frob}} \otimes \cdots \otimes (\mathrm{Sym}^{r_{f-1}})^{\mathrm{Frob}^{f-1}} \otimes \det{}^a$$

$$0 \le r_i \le p-1, \quad 0 \le a \le p^f - 2.$$

(i) If $\bar\rho$ is reducible,

$$\bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & 1 \end{pmatrix} \text{ up to twist.}$$

quotient

generic condition: $0 \le r_i \le p-3$ but not all 0 or $p-3$

ω_f: Serre's fundamental character at level f.

Define (formally)

$$W^{\mathrm{expl}}(\bar\rho) := \left\{ (s_0, \cdots, s_{f-1}) \otimes \theta \;\middle|\; \begin{array}{l} \exists J \subseteq \{0, 1, \cdots, f-1\} \text{ s.t.} \\ \bar\rho|_{I_L} \simeq \begin{pmatrix} \omega_f^{\sum_{j\in J} p^j(s_j+1)} & * \\ 0 & \omega_f^{\sum_{j\notin J} p^j(s_j+1)} \end{pmatrix} \otimes \theta. \\ \text{and comes from a Fontaine-Laffaille mod.} \end{array} \right\}$$

E.g. $f=2$:

$$\bar\rho|_{I_L} \simeq \underbrace{\begin{pmatrix} \omega_2^{(r_0+1)+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix}}_{\text{split}} \simeq \begin{pmatrix} \omega_2^{r_0+2} & * \\ 0 & \omega_2^{p(p-1-r_1)} \end{pmatrix} \otimes \omega_2^{(p-1)+pr_1}$$

$J = \{1\}$

$$\simeq \begin{pmatrix} \omega_2^{p(r_1+2)} & * \\ 0 & \omega_2^{(p-1-r_0)} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)} \simeq \boxed{\begin{pmatrix} 1 & * \\ 0 & \omega_2^{(p-2-r_0)+p(p-2+r_1)} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}}$$

comes from FL iff $* = 0$.

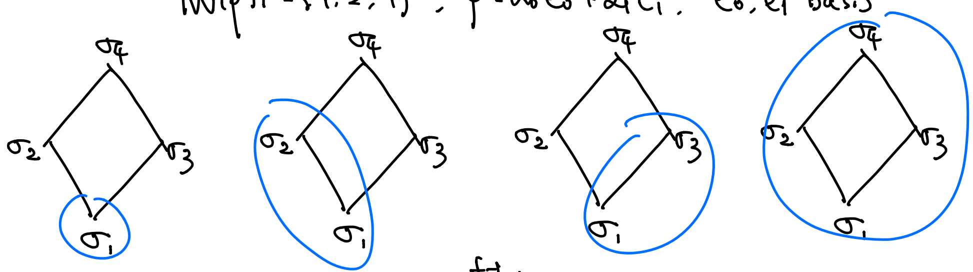
$\exists$ crystalline lift with HT weights $((r_0+2, 0), (0, p-1+r_1))$.

$\omega_2^{p^2-1} = 1$.

$$\underset{\substack{\uparrow \\ \text{split}}}{W^{\text{expl}}}(\bar\rho) = \{ \overset{\sigma_1}{(r_0, r_1)},\ \overset{\sigma_2}{(r_0+1, p-2-r_1) \otimes \det^{p-1+pr_1}},$$
$$\underset{\sigma_3}{(p-2-r_0, r_1+1) \otimes \det^{r_0+p(p-1)}},\ \underset{\sigma_4}{(p-3-r_0, p-3-r_1) \otimes \det^{(r_0+1)+p(r_1+1)}} \}$$

In general, $W^{\text{exp}}(\bar\rho) \subseteq W^{\text{expl}}(\bar\rho^{ss})$. $\sigma_1 \in W^{\text{expl}}(\bar\rho)$, $\sigma_4 \in W^{\text{expl}}(\bar\rho)$ iff $\bar\rho$ splits

$|W(\bar\rho)| = \{1, 2, 4\}$, $\bar\rho = \alpha_0 e_0 + \alpha_1 e_1$. e_0, e_1 basis



When $\bar\rho$ irred, $f=2$, $\bar\rho|_{I_2} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1} p^i(r_i+1)} & 0 \\ 0 & \omega_{2f}^{p^f \cdot \sum_{i=0}^{f-1} p^i(r_i+1)} \end{pmatrix}$

with $J \subseteq \{0, 1, \dots, 2f-1\} \xrightarrow{\text{mod } f} \{0, \dots, f-1\}$ up to twist.

Generic condition: $1 \le r_0 \le p-2$, $0 \le r_i \le p-3$, $i \neq 0$.

$$f=2 \rightsquigarrow W^{\text{expl}}(\bar\rho) = \left\{ \begin{array}{l} (r_0, r_1),\ (r_0-1, p-2-r_1) \otimes \det^?, \\ (p-1-r_0, p-3-r_1) \otimes \det^?,\ (p-2-r_0, r_1+1) \otimes \det^? \end{array} \right\}.$$

§ Concerning about $D_0(\bar\rho)$

Thm (BP) $\exists$ unique f-dim Γ-rep $= GL_2(\mathbb{F}_q)$ $D_0(\bar\rho)$ s.t.

(i) $\mathrm{soc}_\Gamma D_0(\bar\rho) = \bigoplus_{\sigma \in W^{\text{expl}}(\bar\rho)} \sigma$ (Fact: $W^?(\bar\rho) = W(\bar\rho)$.)

(ii) any weight of $W(\bar\rho)$ appears once in $D_0(\bar\rho)$

(iii) $D_0(\bar\rho)$ is max'l w.r.t. (i)(ii).

Moreover, $D_0(\bar\rho)$ is multip one, and $D_0(\bar\rho) = \bigoplus_{\sigma \in W(\bar\rho)} D_{0,\sigma}(\bar\rho)$.

E.g. $f=1$: $\bar\rho = \begin{pmatrix} \omega_1^{r+1} & * \\ 0 & 1 \end{pmatrix}$ non-split,

$$W(\bar\rho) = \{\mathrm{Sym}^r \mathbb{F}^2\} \quad (r_0)$$

$$\text{(i)} \Rightarrow D_0(\bar\rho) \hookrightarrow \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2 \quad \text{(injective envelop)}$$

$$\begin{tikzcd}[column sep=small] & & \mathrm{Sym}^{p-1-r}\otimes \det{}^a \arrow[dr, dash] & \\ D_0(\bar\rho) = \mathrm{Sym}^r \mathbb{F}^2 \arrow[urr, dash] \arrow[drr, dash] & & & \cancel{\mathrm{Sym}^r \mathbb{F}^2} \\ & & \mathrm{Sym}^{p-3-r}\otimes \det{}^{a-1} \arrow[ur, dash] & \end{tikzcd}$$

<u>Rmk</u> $\pi(\bar\rho) \hookrightarrow \mathrm{Inj}_K \left(\bigoplus_{\sigma\in W(\bar\rho)} \sigma\right)[m_F]$.

$\underbrace{GL_2(\mathbb{Q}_p)}$ of ∞-dim'l with Gk-dim $= 4f$.

If $\pi(\bar\rho)^{K_1} \subsetneq (\mathrm{Inj}_K(\cdots))^{K_1} = \mathrm{Inj}_\Gamma \mathrm{Sym}^r \mathbb{F}^2$ $\rightsquigarrow$ a control of Gk $(\pi(\bar\rho))$.

E.g. When irred with $f=1$, $\bar\rho|_{I_L} = \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$.

$\rightsquigarrow W(\bar\rho) = \{\mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-1-r}\otimes \det{}^r\}$

$$D_0(\bar\rho) \hookrightarrow \begin{tikzcd}[column sep=small, row sep=small] & \cancel{\mathrm{Sym}^{p-1-r}} \arrow[dr, dash] & \\ \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r} \\ & \mathrm{Sym}^{p-3-r} \arrow[ur, dash] & \end{tikzcd} \oplus \begin{tikzcd}[column sep=small, row sep=small] & \cancel{\mathrm{Sym}^r} \arrow[dr, dash] & \\ \mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^{p-1-r}} \\ & \mathrm{Sym}^{r-2} \arrow[ur, dash] & \end{tikzcd}$$

§ Patching module / functor

$R_{\bar\rho}[\![x_1,\cdots,x_g]\!]$.

<u>Def</u> A patching module M_∞ is f.g. $R_\infty[GL_2(L)]$-module s.t.

(a) $M_\infty/m_{R_\infty} \simeq \pi(\bar\rho)^\vee$ ⟵ (minimal). M_∞ is projective $S_\infty[\![K]\!]$-mod.

(b) $\forall$ type (ω,τ) $\quad \omega$: of HT wts $(a_i,b_i)_{0\le i\le f-1}$ $\rightsquigarrow R_{\bar\rho}(\omega,\tau)$ Kisin.

$\updownarrow \qquad \tau: I_L \to GL_2(E)$

$\sigma(w,\tau) = K$-rep fin-dim $\supseteq \mathcal{O}$ lattice

Define $M_\infty(\Theta) := \mathrm{Hom}_K(M_\infty, \Theta^\vee)^\vee$ is max'l Cohen-Macaulay
$\circlearrowleft$ (f.g.) $R_\infty(\omega,\tau)$-module.

$R_\infty(\omega,\tau)$
$\|$
$R_\infty \otimes_{R_{\bar\rho}} R_{\bar\rho}(\omega,\tau)$.

Fact If $R_{\bar\rho}(\omega,\tau)=0$, then $M_\infty(\Theta)=0$.

- $M_\infty(-) : \mathcal{O}_E[\![K]\!]$-mod (finite generated over $\mathcal{O}$) $\longrightarrow$ f.g. R_∞-mod.
  is an exact functor.

Cor For $\Theta \subseteq \sigma(\omega,\tau)$ lattice,

$M_\infty(\Theta) \neq 0 \iff M_\infty(\Theta/p\Theta) \neq 0$,

$\iff \exists\, \sigma \in JH(\Theta/p\Theta)$, $M_\infty(\sigma)\neq 0$

JH: Jordan-Hodge factors

Thm $W^?(\bar\rho) = W(\bar\rho)$.

Pf. ① $W^?(\bar\rho) \subseteq W(\bar\rho)$.

Fact (a) in def'n $\Rightarrow M_\infty(\Theta)/m_{R_\infty} \simeq \mathrm{Hom}_K(\underbrace{\Theta, \pi(\bar\rho)}_{})^\vee$
typically, as $(\pi(\bar\rho)^\vee, \Theta^\vee)$.

$\leadsto \sigma \in W^?(\bar\rho) \iff \mathrm{Hom}_K(\sigma, \pi(\bar\rho)) \neq 0 \iff M_\infty(\sigma)\neq 0$.

Assume $\exists\, w=(0,1)$, τ = tame type.

$\sigma \in JH(\overline{\sigma(\tau)})$, $R_{\bar\rho}((0,1),\tau)=0 \Rightarrow M_\infty(\sigma)=0$.

Lemma If $\sigma \notin W(\bar\rho)$ then such a τ always exists.

tame type
- PS type $\rightarrow \sigma(\tau) = \mathrm{Ind}_{B(\mathbb{F}_q)} \tilde\chi_1 \otimes \tilde\chi_2$ (char 0, $(q+1)$-dim)
- cusp $\longrightarrow \sigma(\tau)$ = cusp rep'n $(q-1)$-dim.

$\overline{\sigma(\tau)} = \left(\sigma(\tau)^\circ / p\right)^{ss}$:

$f=1$: $\overline{\sigma(\tau)} = \mathrm{Ind}_B^\Gamma(\omega^a \otimes 1) \rightsquigarrow \overbrace{\mathrm{Sym}^a \mathbb{F}^2 \otimes \mathrm{Sym}^{p-1-a} \otimes \det^a}^{\overset{W(\bar\rho) \text{ for } \bar\rho \text{ irred.}}{\shortparallel}}$

$\mathrm{Sym}^b \notin W(\bar\rho)$.