Lectures on Mod p Langlands Program for GL_2 (2/4)

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<u>Recall</u> $\pi(\bar{\rho})$ = rep'n of $GL_2(L)$

<u>Thm</u> $\mathrm{Soc}_{\underbrace{GL_2(\mathcal{O}_L)}_{= K}} \pi(\bar{\rho}) = \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$ (minimal)

<u>Lemma</u> σ = irred K-rep'n = irred $GL_2(\mathbb{F}_q)$-rep

(i) If $\sigma \in W(\bar{\rho})$ then $\exists$ tame type

$\tau : I_L \longrightarrow GL_2(E) \iff \sigma(\tau)$: rep'n of k

s.t. $\sigma \in JH(\overline{\sigma(\tau)})$, $JH(\overline{\sigma(\tau)}) \cap W(\bar{\rho}) = \{\sigma\}$.

(ii) If $\sigma \notin W(\bar{\rho})$, then $\exists\, \tau$

s.t. $\sigma \in JH(\overline{\sigma(\tau)})$, $JH(\overline{\sigma(\tau)}) \cap W(\bar{\rho}) = \phi$.

Recall that τ is either of principal series type (PS).

or of cusp type

$\rightsquigarrow \sigma(\tau)$ on $GL_2(\mathbb{F}_q)$: either $\underset{\substack{\uparrow \\ q+1}}{\text{PS}}$ or $\underset{\substack{\uparrow \\ q-1}}{\text{cusp}}$.

M_∞ = patching module.

<u>Recall</u> $M_\infty(\sigma(\tau)) \neq 0 \iff M_\infty(\overline{\sigma(\tau)}) \neq 0$

$\iff \exists\, \sigma \in JH(\overline{\sigma(\tau)})$ s.t. $M_\infty(\sigma) \neq 0$.

Proof of Lemma. (i) If $\sigma \notin W(\bar{\rho})$, need $\mathrm{Hom}_K(\sigma, \pi(\bar{\rho})) = 0$ (Gee, 2004 : (i) without Kisin, with the same philosophy)

$\iff M_\infty(\sigma) = 0$.

$\rightsquigarrow$ find τ as in (ii), compute $R_{\bar{\rho}}(\underline{(0,1)}, \tau) = 0$.

(ii) If $\sigma \in W(\bar{\rho})$, need $M_\infty(\sigma) \neq 0$, and

$\dim M_\infty(\sigma)/\mathfrak{m}_{R_\infty} = 1 \iff M_\infty(\sigma)$ is cyclic as an R_∞-mod.

$\rightsquigarrow$ find τ as in (i)

Compute $R_{\bar{\rho}}((0,1),\tau) \neq 0$, a power series ring (regular).

Here $\bar{\rho}$ has a potential BT type τ

$\Rightarrow$ $\bar{r}$ has a lift at p which is of potential BT type τ

By "automorphic lifting"

$\Rightarrow M_\infty(\sigma(\tau)) \neq 0$

$\Rightarrow \exists\, \sigma' \in JH(\overline{\sigma(\tau)})$ s.t. $M_\infty(\sigma') \neq 0$

but σ is the only possibility so that $\sigma = \sigma'$, $M_\infty(\sigma) \neq 0$.

The "minimal" assumption: (= of multiplicity one).

$R_\infty(\tau) = R_\infty \otimes_{R_{\bar{\rho}}^{\mathrm{univ}}} R_{\bar{\rho}}((0,1),\tau)$ is a regular local ring.

& $M_\infty(\sigma)$ is a Cohen-Macaulay module.

(Auslander-Buchsbaum thm:

$$\operatorname{depth} m + \operatorname{proj\,dim} M = \dim R.$$)

Thm $\pi(\bar{\rho})^{K_1} = D_0(\bar{\rho})$.

- $f=1$, $\bar{\rho}$ irreducible. (the case of $\mathbb{Q}_p$).

$$D_0(\bar{\rho}) = \left(\begin{array}{ccc} \mathrm{Sym}^r & \text{---} & \mathrm{Sym}^{p-3-r} \otimes \det^{r-1} \\ \oplus & & | \\ \mathrm{Sym}^{p-1-r} \otimes \det^r & \text{---} & \mathrm{Sym}^{r-2} \end{array}\right) \hookrightarrow \begin{array}{c} \begin{tikzcd}[column sep=small] & \bullet \arrow[dr, dash] & \\ \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{Sym}^r \\ & \cancel{\mathrm{Sym}^{p-1-r}} \arrow[ur, dash] & \end{tikzcd} \\ \oplus \\ \begin{tikzcd}[column sep=small] & \bullet \arrow[dr, dash] & \\ \mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{Sym}^{p-1-r} \\ & \mathrm{Sym}^r \arrow[ur] & \end{tikzcd} \end{array}$$

Choose PS type τ s.t.

$$\sigma(\tau)^\circ/\varpi_E \simeq \boxed{\mathrm{Sym}^r - \mathrm{Sym}^{p-1-r}}$$

To prove that $\mathrm{Hom}_K(\sigma(\tau)^\circ/\varpi_E, \pi(\bar{\rho}))$ is 1-dim'l

$\Leftarrow$ $M_\infty(\sigma(\tau)^\circ/\varpi_E)$ is a cyclic R_∞-module.

$\mathcal{O}_E[\![X,Y]\!]/(XY-p)$ is a regular local ring if $E/\mathbb{Q}_p$ unramified.

[EGS] work with unramified coeff. field

$\Rightarrow$ $M_\infty(\sigma(\tau)^\circ)$ is a cyclic R_∞-mod

multitype deformation ring.

E.g. When $f=1$, reducible non-split, $W(\bar\rho)=\{\mathrm{Sym}^r\}$

$$\bar\rho|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}, \quad D_0(\bar\rho) = \left(\begin{array}{ccc} & & \mathrm{Sym}^{p-3-r} \\ \mathrm{Sym}^r & & \\ & & \mathrm{Sym}^{p-1-r} \end{array} \right) \; \cancel{\mathrm{Sym}^r}$$

Consider τ_1: PS type, reduction Sym^r & Sym^{p-1-r}

τ_2: cusp type, reduction Sym^r & Sym^{p-3-r}.

<u>Lemma</u> $\exists\, \Theta \subseteq \sigma(\tau_1)\oplus\sigma(\tau_2)$ s.t.

$$\Theta/\varpi_E\Theta \simeq \begin{array}{ccccc} & & \bullet & & \\ \bullet & & & & \bullet \\ & & \bullet & & \end{array}$$

<u>Proof</u> $$\big(\underbrace{\mathrm{Proj}_{\mathcal{O}[GL_2(\mathbb{F}_q)]}}_{2p}\,\sigma\big)[\tfrac{1}{p}] = \underset{p+1}{\sigma(\tau_1)} \oplus \underset{p-1}{\sigma(\tau_2)}.$$

<u>Recall</u> H finite grp, σ = irred rep of $H/\mathbb{F}$.

$\rightsquigarrow$ $(\mathrm{Proj}_{\mathcal{O}[H]}\sigma)[\frac{1}{p}]/\varpi = \mathrm{Proj}_{\mathbb{F}[H]}\sigma$

$(\mathrm{Proj}_{\mathcal{O}[H]}\sigma)[\frac{1}{p}]$ (semi-simple) $= \bigoplus_i V_i^{a_i}$

where $a_i = \dim_{\mathbb{F}[H]}(\mathrm{Proj}\,\sigma, \bar V_i) = [\bar V_i : \sigma]$.

Now $\Theta \subseteq \sigma(\tau_1)\oplus\sigma(\tau_2)$

$\twoheadrightarrow \Theta_1 \subseteq \sigma(\tau_1)$ as $\mathcal{O}$-lattices

$\twoheadrightarrow \Theta_2 \subseteq \sigma(\tau_2)$

$$\rightsquigarrow\ 0 \to \Theta \hookrightarrow \Theta_1\oplus\Theta_2 \twoheadrightarrow \overset{p\text{-torsion mod}}{\underset{\sigma}{\boxed{\mathrm{Sym}^r}}} \to 0 \quad \text{(check)}.$$

Taking $M_\infty(-)$:

$$0 \to M_\infty(\Theta) \to M_\infty(\Theta_1) \oplus M_\infty(\Theta_2) \to M_\infty(\sigma) \to 0$$

cyclic modules

Also, $M_\infty(\Theta)$ cyclic $\Leftrightarrow$ $\underset{\mathfrak{p}}{\mathrm{Ann}(M_\infty(\Theta_1))} + \underset{\mathfrak{p}}{\mathrm{Ann}(M_\infty(\Theta_2))} = \underset{\mathfrak{p}}{\mathrm{Ann}(M_\infty(\sigma))}$.

Ex $R_{\bar\rho}((\underline{0,1}), \tau_1) = \mathcal{O}[X,Y]/(X)$, $\quad (\mathcal{O}[X,Y] = R_{\bar\rho}^{univ})$

$R_{\bar\rho}((\underline{0,1}), \tau_2) = \mathcal{O}[X,Y]/(X-p)$

cannot be written as $\mathcal{O}[\![Y]\!]$. need the coordinate.

When $f=2$, need to consider 2 PS types + 2 cusp types

say τ_1, τ_2 & τ_1', τ_2',

such that $\sigma_1 \in JH(\overline{\sigma(\tau_i)}) \cap W(\bar\rho)$, $\sigma_1 \in JH(\sigma(\tau_i')) \cap W(\bar\rho)$.

Assume $\bar\rho$ reducible non-split

$W(\bar\rho) = \{\sigma_1, \sigma_2\}$, $\sigma_1 = (r_0, r_1)$, $\sigma_2 = (p-2-r_0, r_1+1)$

$\rightsquigarrow$ $JH(\overline{\sigma(\tau_1)}) \cap W(\bar\rho) = \{\sigma_1\}$,

$JH(\overline{\sigma(\tau_2)}) \cap W(\bar\rho) = \{\sigma_1, \sigma_2\}$

$JH(\overline{\sigma(\tau_1')}) \cap W(\bar\rho) = \{\sigma_1, \sigma_2\}$

$JH(\overline{\sigma(\tau_2')}) \cap W(\bar\rho) = \{\sigma_1\}$, (by B-M).

E.g. $R_{\bar\rho}^{univ} = \mathcal{O}[\![X_0, Y_0, X_1, Y_1]\!][\![\text{others}]\!]$, $R_{\bar\rho}(?) = R_{\bar\rho}^{univ}/I_?$

$R_{\bar\rho}(\tau_1)$	$R_{\bar\rho}(\tau_2)$	$R_{\bar\rho}(\tau_1')$	$R_{\bar\rho}(\tau_2')$
(X_0, X_1)	(X_0-p, X_1Y_1-p)	(X_0, X_1Y_1-p)	(X_0-p, X_1)

Take $\Gamma = \mathbb{F}[GL_2(\mathbb{F}_q)] = \mathbb{F}[\![K]\!]/\mathfrak{m}_{K_1}$, $\tilde\Gamma := \mathbb{F}[\![K]\!]/\mathfrak{m}_{K_1}^2$

with $\underbrace{\mathbb{F}[\![K_1]\!]}_{\text{pro-}p\text{ grp}} \subseteq \mathbb{F}[\![K]\!]$, local ring $\mathfrak{m}_{K_1} = (k_1 - 1)$.

$\rightsquigarrow$ $\pi(\bar\rho)^{K_1} = \pi(\bar\rho)[\mathfrak{m}_{K_1}]$ (killed by $\mathfrak{m}_{K_1}$)

$$\pi(\bar\rho)[\mathfrak{m}_{K_1}^2] \subsetneq \pi(\rho_2)^{K_2}.$$

$$\hookrightarrow (\mathrm{Inj}_K \mathrm{Sym}^r)[\mathfrak{m}_{K_1}^2].$$

- Motivation: $f=1$, $\bar\rho$ = red non-split

$$\pi(\bar\rho)^{K_1} = D_0(\bar\rho) = \begin{tikzcd}[row sep=small] & \mathrm{Sym}^{p-3-r} \arrow[dr, dash] & \\ \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r} \\ & \mathrm{Sym}^{p-1-r} \arrow[ur, dash] & \end{tikzcd}$$

$$\pi(\bar\rho)|_K \hookrightarrow \mathrm{Inj}_K \mathrm{Sym}^r \longrightarrow \mathrm{Sym}_K \mathrm{Sym}^r.$$

- Dual: $\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \xrightarrow{x} \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \longrightarrow \pi(\bar\rho)^\vee \longrightarrow 0$

$$\rightsquigarrow \underbrace{\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee/x}_{\text{Gk dim}=3 \text{ (fix control character)}} \twoheadrightarrow \pi(\bar\rho)^\vee \quad \text{(expect dim} = 1)$$

- Aim: control $\mathrm{Gk}(\pi(\bar\rho))$

$\Rightarrow \mathrm{Gk}(\pi(\rho)^\vee) \le \mathrm{Gk}(\mathrm{Proj}(\mathrm{Sym}^r)^\vee/x)$. | To prove: $\mathrm{Gk}(\mathrm{Proj}_K(\mathrm{Sym}^r)^\vee/(x,y)) = 1$.

<u>Recall</u> $0 \to V^{K_1} \to V \to V/V^{K_1} \to 0$

$$\Rightarrow \quad 0 \to V^{K_1} \xrightarrow{=} V^{K_1} \to (V/V^{K_1})^{K_1} \xrightarrow{=} H^1(K_1, V^{K_1}) \to 0 \qquad (f=1)$$

$$\underbrace{(\mathrm{Sym}^2\mathbb{F}^2 \otimes \det^{-1})}_{SL_2} \otimes V^{K_1} = H^1(K_1,\mathbb{F}) \otimes V^{K_1}$$

$$H^1(K_1,\mathbb{F}) = \underbrace{\mathrm{Hom}(K_1/Z_1, \mathbb{F})}_{3\text{-dim'l } \mathbb{F}\text{-v.s.}} \circlearrowleft \Gamma\text{-action}$$

Get $0 \to \mathrm{Inj}_\Gamma \mathrm{Sym}^r \to (\mathrm{Inj}_K \mathrm{Sym}^r)[\mathfrak{m}_{K_1}^2] \to \underbrace{(\mathrm{Sym}^2 \otimes) \otimes \mathrm{Inj}_\Gamma \mathrm{Sym}^r}_{\circlearrowleft\, \Gamma.} \to 0$

$(2 < r < p-3)$

<u>Thm</u> $\pi(\bar\rho)[\mathfrak{m}_{K_1}^2]$ is of "multiplicity one".

(In particular, $\forall \sigma \in W(\bar\rho)$,
$$\begin{tikzcd}[row sep=small] & \cdot \arrow[r, dash] & \cancel{\mathrm{Sym}^r} \\ \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \\ & \cdot \arrow[r, dash] & \cancel{\mathrm{Sym}^r} \end{tikzcd}$$
σ occurs once in $\pi(\bar\rho)[\mathfrak{m}_{K_1}^2]$)

Thus,

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \mathrm{Sym}^{p+2} \arrow[r, dash] & \mathrm{Sym}^{p-1-r} \\
 & \mathrm{Sym}^{p-3-r} \arrow[ur, dash] \arrow[dr, dash] & & \\
\pi(\bar{p})[\mathfrak{m}_{K_1}^2] = \mathrm{Sym}^r \arrow[ur, dash] \arrow[dr, dash] & & \cancel{\mathrm{Sym}^r \oplus \mathrm{Sym}^r} & \\
 & \mathrm{Sym}^{p-1-r} \arrow[ur, dash] \arrow[dr, dash] & & \\
 & & \mathrm{Sym}^{p-2} \arrow[r, dash] & \mathrm{Sym}^{p+1-r}
\end{tikzcd}$$

<u>Proof</u>. $(\mathrm{Sym}^2 \mathbb{F}^2 \otimes \det^{-1}) \otimes \mathrm{Proj}_\Gamma \mathrm{Sym}^r$

$$\xrightarrow{\text{lift to char } 0} \underbrace{(\mathrm{Sym}^2 \mathcal{O}^2 \otimes \det^{-1})}_{\text{Alg rep'n}} \otimes \underbrace{(\mathrm{Proj}_{\mathcal{O}[\Gamma]} \mathrm{Sym}^r)[\tfrac{1}{p}]}_{= \text{tame type direct sum.}}$$

Use deformation ring $(-1,2)$ of tame type, multitype.

$\rightsquigarrow$ $(\mathrm{Proj}_{\mathcal{O}[\tilde{\Gamma}]} \sigma)[\frac{1}{p}]$ smooth $\tilde{\Gamma}$-rep'n, $\tilde{\Gamma}$ finite algebra.

Attempt: $GL_2(\mathbb{Q}_p)$ + Paskunas:

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{Proj}(\mathrm{Sym}^r)^\vee \arrow[dr, "x"] & & & \\
0 \arrow[r] & \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \arrow[ur, "-y"] \arrow[dr, "x"'] & & \mathrm{Proj}_K(\mathrm{Sym}^r)^\vee \arrow[r] & \mathrm{coker} \arrow[r] & 0 \\
 & & \mathrm{Proj}(\mathrm{Sym}^r)^\vee \arrow[ur, "y"'] & & \pi(\bar{p})^\vee \arrow[u] &
\end{tikzcd}$$

§ Iwahori subgroup

$$I = \begin{pmatrix} * & * \\ p* & * \end{pmatrix} \supseteq I_1 = \text{Sylow pro-}p \text{ Iwahori}.$$

$\rightsquigarrow$ $\mathbb{F}[\![I_1/Z_1]\!]$ Iwasawa algebra,

a $3f$-dim'l noetherian domain.

I_1/Z_1 is <u>not uniform</u>: $\mathrm{Hom}(I_1/Z_1, \mathbb{F}) = 2f$. ($K_1/Z_1$ unramified)

corresp. to poly ring of $3f$ variables.

$\mathrm{gr}_{\mathfrak{m}_{I_1}}(\mathbb{F}[\![I_1/Z_1]\!])$ not commutative.

Thm $\mathrm{gr}_{m_{I_1}}(\mathbb{F}[[I_1/Z_1]]) = U(\mathfrak{g})$

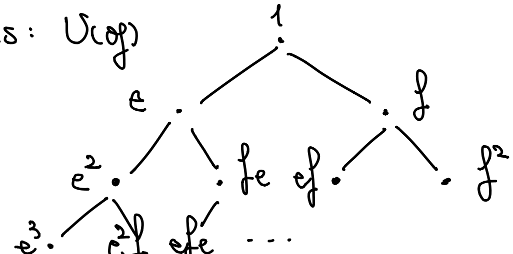
where $\mathfrak{g} = \bigoplus_{i=0}^{f-1} \mathbb{F}e_i^{(1)} \oplus \mathbb{F}f_i^{(1)} \oplus \mathbb{F}h_i^{(2)}$, $h_i \in Z(\mathfrak{g})$.

s.t. $[e_i, f_i] = h_i$, $[e_i, h_i] = [f_i, h_i] = 0$, $[e_i, f_j] = 0$ $(i \neq j)$.

When $f=1$, $\mathfrak{g} = \mathbb{F}e \oplus \mathbb{F}f \oplus \mathbb{F}h$, $h = ef - fe$.

and $e(ef - fe) = (ef - fe)e$.

PBW basis: $U(\mathfrak{g})$



Take $H \curvearrowright \mathfrak{g}$, $H \ni \alpha_i : \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \longmapsto ad^{-1}$

$\rightsquigarrow$ $g \cdot e_i = \alpha_i(g) \cdot e_i$, $g \cdot f_i = \alpha_i^{-1}(g) \cdot f_i$, $g \cdot h_i = h_i$.

with $e_i \longleftrightarrow \begin{pmatrix} 1 & [x_i] \\ & 1 \end{pmatrix} - 1$.

On the H-rep'n page,

Facts (i) h_i is central in $U(\mathfrak{g})$, $U(\mathfrak{g})/(h_0, \dots, h_{f-1}) \simeq \mathbb{F}[e_i, f_i]_{0 \le i \le f-1}$.

poly ring of Krull dim $= 2f$.

(ii) $U(\mathfrak{g})/(e_i f_i, f_i e_i)$ is a commutative noetherian ring of Krull dim $= f$.

Thm Let π = adm smooth rep'n of I/Z_1.

Assume $\forall x$ in $\mathrm{soc}_I(\pi)$, $[\pi[m_{I_1}^3] : x] \overset{(*)}{=} [\pi[m_{I_1}] : x]$ then $\mathrm{GK}\text{-}\dim(\pi) \le f$.

Further condition (*): $\pi^\vee$ is f.g. $\mathbb{F}[\![I_1/Z_1]\!]$-mod $\circlearrowleft$ H-action

$gr(\pi^\vee)$ is f.g. $gr(I_1/Z_1)$-mod.

$\Rightarrow$ $(e_if_i, f_ie_i) \in \mathrm{Ann}_{U(\mathfrak{g})}(gr(\pi^\vee))$.

i.e. $gr(\pi^\vee)$ is f.g. module over $\underbrace{\mathbb{F}[e_i, f_i]/(e_if_i)}_{\text{Krull-dim} = f}$.

$\Rightarrow$ $GK\text{-}dim(gr(\pi^\vee)) \le f$

$\Rightarrow$ $GK(\pi^\vee) \le f$.

Thm $GK(\pi(\bar\rho)) \le f$

Proof Known: $\pi(\bar\rho)[\mathfrak{m}^2_{K_1}] \circlearrowleft K$ is of multiplicity one.

$\Downarrow$ Frob-reciprocity.

I-rep'n. $\pi(\bar\rho)[\mathfrak{m}^3_{I_1}]$ is of multiplicity one. (*)

$\Downarrow$

$GK(\pi(\bar\rho)) \le f$.