Lectures on Mod p Langlands Program for GL_2 (3/4)

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Recap $\pi(\bar{\rho})$: $GL_2(L)$-rep'n

with GK-dim of $\pi(\bar{\rho}) \leq f$ (+ Gee-Newton) $\Rightarrow$ GK dim $= f$

Take $\mathbb{F}[\![I_1/Z_1]\!]$ enveloping algebra.

$$\rightsquigarrow \operatorname{gr}_{m_{I_1}}(\mathbb{F}[\![I_1/Z_1]\!]) \simeq U(\mathfrak{g}), \quad \mathfrak{g} = \langle e_i, f_i, h_i \rangle_{0 \leq i \leq f-1}$$

$$\cup\!\!\!\;| \qquad J = (e_i f_i, f_i e_i)_{0 \leq i \leq f-1}.$$

Theorem $\operatorname{gr}(\pi(\bar{\rho})^\vee)$ is annihilated by J.

Def'n Category $\mathcal{C} := \{\pi : \text{adm sm s.t. } \operatorname{gr}_{m_{I_1}}(\pi^\vee) \text{ is killed by } J^n, \text{ for some } n\}$.

$\rightsquigarrow \forall \pi \in \mathcal{C}$, $GK(\pi) \leq f$.

Remark: $\pi(\bar{\rho}) = S(U^v, \mathbb{F})[m_{\mathbb{J}}]$, $\mathbb{J}$ = Hecke algebra.

$\rightsquigarrow$ looking at $S(U^v, \mathbb{F})[m_{\mathbb{J}}^k] \in \mathcal{C}$.

Today To define generalized Colmez's functor for $\pi \in \mathcal{C}$.

§ Colmez's functor

$\mathbb{D}$: {adm rep'n of $GL_2(\mathbb{Q}_p)$ of finite length} $\longrightarrow$ {étale (φ, Γ)-mod}
(torsion coeffs) (torsion coeffs)

Recall $\pi \in$ LHS over $\mathbb{F}$, $P^+ = \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ semigroup, $\mathbb{Z}_p \setminus \{0\} = p^{\mathbb{N}} \times \mathbb{Z}_p^\times$

$\cup\!\!\!\;|$ $\qquad\qquad \updownarrow$

$W \qquad (\varphi, \Gamma)\text{-mod} / \mathbb{F}[\![X]\!] = \mathbb{F}[\![\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}]\!]$.

$\|$

fin-dim'l $GL_2(\mathbb{Z}_p)$-rep'n, generating π.

Define $I_w^+(\pi) := \langle P^+, w\rangle \subseteq \pi$

$\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, \Gamma \circlearrowright$

$I_w^-(\pi) := \begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix} \langle P^+, w\rangle \subseteq \pi$

$\begin{pmatrix} p^{-1} & 0 \\ 0 & 1\end{pmatrix}, \Gamma \circlearrowright$

$\rightsquigarrow G = P^+ \cdot GL_2(\mathbb{Z}_p) \cup \begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix} P^+ GL_2(\mathbb{Z}_p)$

$\rightsquigarrow 0 \to I^+(\pi) \cap I^-(\pi) \to I^+(\pi) \oplus I^-(\pi) \to \pi \to 0$

Dual $\mathbb{D}(\pi) := \underbrace{I_w^+(\pi)^\vee}_{(\psi,\Gamma)\text{-mod, with } \pi^\vee \twoheadrightarrow I_w^+(\pi)^\vee} \otimes_{\mathbb{F}[[X]]} \mathbb{F}((x))$

Colmez: proved that $\mathbb{D}(\pi)$ is an étale (φ,Γ)-mod.

Define $D^+(\pi) := \{\mu \in \pi^\vee \mid \mu|_{I^-(\pi)} = 0\} \hookrightarrow \pi^\vee$

$\rightsquigarrow D^+(\pi) \otimes_{\mathbb{F}[[X]]} \mathbb{F}((x)) \hookrightarrow \pi^\vee \otimes_{\mathbb{F}[[X]]} \mathbb{F}((x)) \to I_w^+(\pi)^\vee \otimes_{\mathbb{F}[[X]]} \mathbb{F}((x))$

$(\varphi,\Gamma), \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} \circlearrowright$

Generally $D^+(\pi) \hookrightarrow \pi^\vee \twoheadrightarrow D^\natural(\pi)$ (D-nature)

$(-) \otimes \mathbb{F}((x))$ becomes an isom (difference is $(\underbrace{I^+(\pi) \cap I^-(\pi)}_{\text{fin-dim'l (only for } GL_2(\mathbb{Q}_p))})^\vee$).

Fact $I^+(w)^{\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1\end{pmatrix}}$ is finite dim'l $\mathbb{F}$-v.s. (only for $GL_2(\mathbb{Q}_p)$)

$\Rightarrow I^+(w)^\vee$ is a f.g. $\mathbb{F}[[X]]$-module

$\Rightarrow \mathbb{D}(\pi)$ is a finite-rank (φ,Γ)-module.

Key All irred rep'ns of $GL_2(\mathbb{Q}_p)$ are of finite presentation!

(FALSE for ss. $GL_2(L)$, $L \neq \mathbb{Q}_p$).

Theorem $\mathbb{D}$ is an exact functor $\xrightarrow{\text{Fontaine}}$ $\mathbb{V} : \pi \mapsto$ Galois rep'n. s.t.

(1) $\mathbb{V}(\chi \circ \det) = 0$,

(2) $\mathbb{V}(\mathrm{Ind}_B^G \chi_1 \otimes \chi_2 \omega^{-1}) = \chi_2$,

(3) $\mathbb{V}(\text{s.s.}) = $ 2-dim'l irred $\bar{\rho}$.

<u>Generalization</u> problem lies in $I^{+}(\pi)^{\begin{pmatrix}1 & \mathcal{O}_L\\ 0 & 1\end{pmatrix}}$ is ∞-dim'l
(if $L \neq \mathbb{Q}_p$, π is s.s.).

<u>Breuil's version (2015)</u>

$\mathrm{tr}: \mathcal{O}_L \to \mathbb{Z}_p$ trace map, $N_0 = \begin{pmatrix}1 & \mathcal{O}_L\\ 0 & 1\end{pmatrix} \trianglerighteq N_1 = \begin{pmatrix}1 & \ker(\mathrm{tr})\\ 0 & 1\end{pmatrix}$.
$\rightsquigarrow N_0/N_1 \simeq \mathbb{Z}_p$.

π : adm. rep'n of $GL_2(L)$.

$\pi^{N_1} \subseteq \pi$ carries $\begin{cases} \text{action of } \begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix} \\ \text{action of } \begin{pmatrix}\mathbb{Z}_p\setminus\{0\} & 0\\ 0 & 1\end{pmatrix}\end{cases}$.

$$x \cdot v = \sum_{n_1 \in N_1 / \begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} N_1 \begin{pmatrix}x^{-1} & 0\\ 0 & 1\end{pmatrix}} n_1 \begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} v \in \pi^{N}, \quad x \in \mathbb{Z}_p\setminus\{0\}, \ v \in \pi^{N_1}.$$

<u>Fact</u> tr is $\mathbb{Z}_p$-linear

<u>Def'n</u> A subspace $W \subseteq \pi^{N_1}$ is called <u>admissible</u> if W is stable under P^{+}-action, and $W^{\begin{pmatrix}1 & \mathbb{Z}_p\\ 0 & 1\end{pmatrix}}$ is of finite dim.

An analogue of $I_{w}^{+}(\pi)$.

$\Rightarrow W^{\vee}$ is f.g. $\mathbb{F}[\![X]\!]$-mod, with (φ,Γ)-mod structure

<u>Lemma</u> If W is admissible, then $\mathbb{F}((X)) \otimes_{\mathbb{F}[\![X]\!]} W^{\vee}$ is étale (φ,Γ)-module.

<u>Def'n</u> (Breuil) $\mathbb{D}(\pi) := \varprojlim_{\substack{W \subseteq \pi^{N_1} \\ \text{adm}}} (\mathbb{F}((X)) \otimes_{\mathbb{F}[\![X]\!]} W^{\vee})$ pro-étale (φ,Γ)-mod.

<u>Breuil's work</u> (1) compute $\mathbb{D}(\pi)$ if π is not ss.
(2) $\mathbb{D}$ left-exact, and exact on non-ss rep'ns.

Not known (1) $\{W \text{ adm}\}$ is non-empty?

(2) $\mathbb{D}(\pi)$ is finite generated $/ \mathbb{F}((X))$?

(3) $\mathbb{D}(-)$ is exact?

Theorem (Hu-Wang) On category $\mathcal{C}$,

$\mathbb{D}$ is exact & $\mathbb{D}(\pi)$ is f.g. over $\mathbb{F}((X))$.

· For $\pi(\bar\rho) \in \mathcal{C}$, $\mathbb{D}(\pi(\bar\rho))$ is explicitly determined.

§ The ring A

$$\mathbb{F}[\![N_0]\!] \simeq \mathbb{F}[\![Y_0, \cdots, Y_{f-1}]\!], \text{ where } Y_i = \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{p^i} \begin{pmatrix} 1 & [\bar\lambda] \\ 0 & 1 \end{pmatrix} \in m_{N_0}.$$

$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ (= $\mathbb{F}[\![N_0]\!]$), $\cup$ m_{N_0}

Teichmüller lifting with $\sum_{\lambda \in \mathbb{F}_q} \lambda^{-p^i} = 0$.

m_{N_0}-adic filtration

$$\forall a \in N_0, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} - 1 \in m_{N_0}.$$

(Y_i eigenvector for the action of $\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}$).

Let $S := \{(Y_0, \cdots, Y_{f-1})^n : n \geq 0\}$ multip subset

$\mathbb{F}[\![N_0]\!]_S$: extend a filtration.

$$\deg \frac{h}{(Y_0 \cdots Y_{f-1})^n} = \deg h - nf.$$

Def'n $A = \widehat{\mathbb{F}[\![N_0]\!]_S}$ (in general, $\hat{M} = \varprojlim_n M / \mathrm{Fil}_n M$).

$$\rightsquigarrow \mathrm{gr}(A) = \mathrm{gr}(\mathbb{F}[\![N_0]\!]_S) \simeq \underbrace{\mathrm{gr}(\mathbb{F}[\![N_0]\!])}_{= \mathbb{F}[y_0, \cdots, y_{f-1}]}[(y_0, \cdots, y_{f-1})^{-1}]$$

y_i = principal part of Y_i.

On A: $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ on N_0

$\Rightarrow \varphi : \mathbb{F}[\![N_0]\!] \longrightarrow \mathbb{F}[\![N_0]\!]$ finite flat of degree q.

Check: $\varphi(Y_i) = Y_{i-1}^p$, φ extends to $\mathbb{F}[\![N_0]\!]_S$, and then extends to A.

$\mathcal{O}_L^\times$-action: $\gamma_a : \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ on N_0

doesn't preserve S.

But γ_a extends to A, i.e. $\gamma(y_0, \cdots, y_{f-1}) \notin \mathbb{F}[\![N_0]\!]_S$, $\in A^\times$

<u>Def'n</u> A $(\varphi, \mathcal{O}_L^\times)$-mod over A is a f.g. A-mod with commuting $(\varphi, \mathcal{O}_L^\times)$-action.

<u>Prop</u> If M is f.g. A-mod with an $\mathcal{O}_L^\times$-semilinear action, then M is finite free as an A-module. (Colmez: $A = \mathbb{F}((X))$).

<u>Proof</u> <u>Step 1</u> An ideal of A is stable under $\mathcal{O}_L^\times$ is either 0 or A itself.

$\Rightarrow$ if M is a torsion A-mod,

then $(\mathrm{Ann}_A(M) \neq 0 \Rightarrow M = 0)$.

<u>Step 2</u> (Double duality spectral sequence):

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{Ext}_A^q(M, A), A) \Rightarrow H^{p-q}(M) = \begin{cases} M, & p-q=0 \\ 0, & p-q \neq 0 \end{cases}.$$

If $p \neq 0$ or $q \neq 0$, $E_2^{p,q}$ is A-torsion. (Auslander regular)

$\Rightarrow E_2^{p,q} = 0$

We obtain
$$\begin{cases} \mathrm{Hom}(\mathrm{Hom}(M, A), A) \simeq M & ① \\ \mathrm{Ext}_A^p(\underbrace{\mathrm{Hom}(M, A)}_{M'}, A) = 0, \quad \forall p > 0 & ② \end{cases}$$

② $\Rightarrow M'$ is a projective module $\Rightarrow M$ proj.

<u>Step 3</u> Lütkebohmert (1977)

§ The functor $\mathcal{C} \to \{\text{étale } (\varphi, \mathcal{O}_K^\times)\text{-mod}\}$

$\pi \in \mathcal{C}$, $\pi^\vee$ is f.g. $\mathbb{F}[\![I_1/Z_1]\!]$-module (but not f.g. over $\mathbb{F}[\![N_0]\!]$).

$$(\pi^\vee)_S = \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee .$$

Key give "tensor product filtration on $(\pi^\vee)_S$"

- $\mathbb{F}[\![N_0]\!]_S$ the usual one   but $\exists\ \mathbb{F}[\![N_0]\!] \to \mathbb{F}[\![I_1]\!]$.

To kill the "negative" part →

- $\pi^\vee$: $\mathfrak{m}_{I_1}$-adic filtration (!) NOT $\mathfrak{m}_{N_0}$-adic filtration,

Define $D_A(\pi) := \widehat{(\pi^\vee_S)} = \varprojlim_n \pi^\vee_S / \mathrm{Fil}^n \pi^\vee_S$.

Rmk $\mathbb{Q}_p$ case: $\mathbb{F}((x)) \widehat{\otimes}_{\mathbb{F}[\![x]\!]} \pi^\vee = \underbrace{\mathbb{F}((x)) \otimes_{\mathbb{F}[\![x]\!]} I^+(\pi)^\vee}_{\text{Colmez's def'n}} \circlearrowleft I_1\text{-action, positive.}$

(Obstruction: $I(\pi)^\vee$ is not f.g. over $\mathbb{F}((x))$).

Reason: $\mathbb{F}((x)) \widehat{\otimes}_{\mathbb{F}[\![x]\!]} I(\pi)^\vee = 0$ b/c $\mathrm{Fil}^n M = \mathrm{Fil}^{n+1} M = \cdots$ from some n.

$$\underset{\deg = 0}{v} \otimes \underset{\deg=-1}{w} = \frac{v}{y} \otimes \underset{\deg \text{"}p-1\text{"}}{yw} \qquad \text{on } I(\pi)^\vee: \deg yw > 1 + \deg w \text{ under } \mathfrak{m}_{I_1}\text{-adic filtration}$$

$$= \frac{v}{y^2} \otimes y^2 w \qquad \text{on } I^+(\pi)^\vee: \deg(y/w) = 1 + \deg w .$$

Exactness: $0 \to \pi_1 \to \pi \to \pi_2 \to 0$

$$0 \to \pi_2^\vee \to \pi^\vee \to \pi_1^\vee \to 0$$

Check: $\widehat{(-)_S} \rightsquigarrow$ exactness of $D_A(-)$.

Finiteness: $D_A(\pi)$ is a finite (free) A-mod for $\pi \in \mathcal{C}$

Pf. It suffices to show $\mathrm{gr}(D_A(\pi))$ is f.g. $\mathrm{gr}(A)$-module

$$\mathrm{gr}(D_A(\pi)) = \mathrm{gr}_{\mathfrak{m}_{I_1}}(\pi^\vee)[(y_0 \cdots y_{f-1})^{-1}]. \qquad \mathrm{gr}(A) = \mathbb{F}[y_0, \dots, y_{f-1}][(y_0, \dots, y_{f-1})^{-1}]$$

Assume it is killed by J (WLOG)

$\Rightarrow$ f.g. module over $\mathbb{F}[y_i, z_i]/(y_i z_i)$

Key $(\mathbb{F}[y_i, z_i]/(y_i z_i))[(y_0, \dots, y_{f-1})^{-1}] \simeq \mathbb{F}[y_i^{\pm 1}, 0 \le i \le f-1] = \mathrm{gr}(A)$. □

Action $\mathcal{O}_K^\times$ acts on $D_A(\pi)$

φ-action $\circlearrowleft \widehat{\pi_S}$

(small issue) $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$-action on π will become a ψ-action on $\pi^\vee$.

Define $\psi: \pi^\vee \to \pi^\vee$, $\mu \longmapsto \mu \circ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

which satisfies $v \in \pi^\vee$, $a \in \mathbb{F}[\![N_0]\!]$, $\psi(\varphi(a)\mu) = a \cdot \psi(\mu)$.

ψ extends to $(\pi^\vee)_S$: $\dfrac{\mu}{(Y_0, \dots, Y_{f-1})^{pn}} \longmapsto \dfrac{\psi(\mu)}{(Y_0, \dots, Y_{f-1})^n}$

$\rightsquigarrow$ ψ extends to $D_A(\pi) \to D_A(\pi)$

- How to get an action of φ?

Rmk M over (A, φ). Then

$\exists$ étale φ-action on $M \Leftrightarrow$ $A \otimes_{A,\varphi} M \xrightarrow{\sim} M$ isom

$a \otimes m \longmapsto a\varphi(m)$.

It is equivalent to: $\exists$ ψ-action $\psi: M \to M$ satisfying

(i) $\psi(\varphi(a)m) = a \cdot \psi(m)$

(ii) an isom $M \xrightarrow{(*)} A \otimes_{\varphi, A} M$

$m \longmapsto \underbrace{\sum_{n \in N_0/N_0^p}}_{\text{a basis of } A \text{ over } \varphi(A).} \delta_n \otimes \psi(\delta_n^{-1} m)$, $\delta_n \in [n] \in \mathbb{F}[\![N_0]\!] \subseteq A$.

But In general, not able to prove the map

$D_A(\pi) \xrightarrow{(*)} A \otimes_{A,\varphi} D_A(\pi)$ is an isom.

Prop $\exists$ a max'l quotient of $D_A(\pi)$, denoted by $D_A(\pi)^{et}$

s.t. $D_A(\pi)^{et}$ is stable under ψ, $\mathcal{O}_K^\times$, and $D_A(\pi)^{et} \xrightarrow{\sim} A \otimes_{\varphi, A} D_A(\pi)^{et}$

is an isom.

$\Rightarrow D_A(\pi)^{et}$ is étale $(\varphi, \mathcal{O}_L^\times)$-mod.

(This $(-)^{et}$ is formal, doesn't depend on $D_A(\pi)$).

$\rightsquigarrow$ Get a functor $D_A(-)^{et} = (\widehat{\pi_S^\vee})^{et}$, with exactness.

§3 Computation on $D_A(\pi(\bar\rho))$.

Relation to Breuil's version $(\varphi, \mathbb{Z}_p^\times)$-module

$$\mathbb{F}[\![N_0]\!] \xrightarrow{tr} \mathbb{F}[\![\mathbb{Z}_p]\!] : Y_0, \dots, Y_{f-1} \longmapsto X$$

$$\rightsquigarrow \widehat{\mathbb{F}[\![N_0]\!]_S} \longrightarrow \mathbb{F}((X))$$

Given $\pi \in \mathcal{C}$, this gives $D_A(\pi) \to \mathbb{D}_{Breuil}(\pi)$

In fact $D_A(\pi)^{et}/(Y_i - Y_0)_{1\le i\le f-1} \simeq \mathbb{D}_{Breuil}(\pi)$. $\leftarrow$ need étaleness (but freeness doesn't need ét). $\Updownarrow$ with $\mathcal{O}_L^\times$-semilin action,

Thm $\mathbb{D}_{Breuil}$ is exact.

Proof. $D_A(\pi)^{et}$ is a free A-module. $\square$

Compute: $\mathbb{D}_{Breuil}(\pi(\bar\rho))$ finite étale (φ, Γ)-module.

Thm (Conjecturally) $\mathbb{V}_{Breuil}(\pi(\bar\rho)) = \boxed{\mathrm{ind}_L^{\otimes \mathbb{Q}_p} \bar\rho}$ tensor induction. $= \mathrm{ind}_{G_L}^{\otimes G_{\mathbb{Q}_p}} \bar\rho$, of $\dim = 2^f$.

Recall $H < G$ of index n, $G = \coprod_{1\le i\le n} g_i H$

$(\rho, V) =$ fin-dim'l rep'n of H

$\rightsquigarrow$ ① $\mathrm{ind}_H^{\otimes G} \rho = \bigotimes_{i=1}^n (g_i \otimes V)$

② $g'(g_i \otimes v_i) := g_{i'} \otimes (g_{i'}^{-1} g' g_i) v_i$, if $g' g_i \in g_{i'} H$.

with $\dim = (\dim V)^n$.

Recall $\bar\rho$ semisimple $\Rightarrow \# W(\bar\rho) = 2^f$

$(2^f = \# \mathrm{Soc}_{GL_2(\mathcal{O}_L)} \pi(\bar\rho).)$

Step 1 upper bound of $\dim \mathbb{V}_{Breuil}(\pi(\bar\rho)) = \mathrm{rank}_A (D_A(\pi(\bar\rho))^{et}) \le \mathrm{rank}_A D_A(\pi(\bar\rho))$.

Back to proof of $D_A(\pi(\bar\rho))$ f.g.

$\Rightarrow \mathrm{gr}(D_A(\pi(\bar\rho)))$ f.g. over $\mathrm{gr}(A) = \mathbb{F}[y_i^{\pm 1}]$.

Generally # of generators of $D_A(\pi)$

$\le$ # of generators of $\mathrm{gr}(D_A(\pi))$.

Over $\mathbb{F}[y_i, z_i]/(y_i z_i)$: minimal prime ideal $\mathfrak{p}$

$$\mathfrak{p}_0 = (y_0, \dots, y_{f-1})$$

then $\mathrm{rank}_A D_A(\pi(\bar\rho)) \le m_{\mathfrak{p}_0}(\mathrm{gr}\, D_A(\pi(\bar\rho)))$ multiplicity

$$m_{\beta_0}(M) = \mathrm{length}_{A_{\beta_0}}(M_{\beta_0})$$

Can compute $\mathrm{gr}(\pi(\bar{\rho})^\vee)$ with multiplicity at β_0.

<u>Prop</u> $m_{\beta_0}(\mathrm{gr}(\pi(\bar{\rho}))^\vee) \leq 2^f$.

<u>Step 2</u> Find inside π^{N_1}, an adm $W \subseteq \pi^{N_1}$ s.t.

$$\mathbb{F}((X)) \otimes_{\mathbb{F}[[X]]} W^\vee \simeq (\varphi, \mathbb{Z}_p^\times)\text{-mod of } \mathrm{ind}_L^{\otimes \mathbb{Q}_p} \bar{\rho}.$$

<u>Recall</u> $D_{\mathrm{Breuil}}(\pi) = \varinjlim_{\substack{W \subseteq \pi^{N_1} \\ \mathrm{adm}}} \mathbb{F}((X)) \otimes_{\mathbb{F}[[X]]} W^\vee$

get $\dim N_{\mathrm{Breuil}}(\pi(\bar{\rho})) \geq 2^f$

Byproduct: $D_A(\pi(\bar{\rho})) = D_A^{\text{ét}}(\pi(\bar{\rho}))$.