# Lectures on Mod $p$ Langlands Program for $GL_2$ (4/4)

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**Recall** Defined generalized Colmez's functor.

Also Breuil's version:

$$\mathbb{D}_B : \pi \longmapsto \text{proétale } (\varphi,\Gamma)\text{-module}$$

If $\pi \in \mathcal{C}$, then obtain étale $(\varphi,\Gamma)$-module killed by $J^n$.

(as $\dim \mathbb{V}_B(\pi) \le m_{\mathfrak{p}_0}(gr(\pi^\vee))$. $\mathfrak{p}_0 = (y_0, \cdots, y_{f-1})$).

**Theorem** $\mathbb{V}_B(\pi(\bar\rho)) = \mathrm{Ind}_{\mathbb{1}}^{\otimes \mathbb{Q}_p} \bar\rho$ ($\dim = 2^f$).

E.g. For $\dim \mathbb{V}_B(\pi(\bar\rho)) \le 2^f$ i.e. $m_{\mathfrak{p}_0}(gr(\pi^\vee)) \le 2^f$.

- $f=1$, $\bar\rho$ = reducible split, $\pi(\bar\rho) = \pi_0 \oplus \pi_1$ (principal series)

$\pi_0^{I_1} = 2$-dim'l, $\chi_0 \oplus \chi_0^s$ conjugation by $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$.

$\pi_1^{I_1} = 2$-dim'l, $\chi_1 \oplus \chi_1^s$

Serre weights: $\mathrm{Sym}^r \mathbb{F}^2 = \sigma_0$, $\mathrm{Sym}^{p-3-r} \otimes \det^{r+1} = \sigma_1$. $\longleftrightarrow \chi_1^s : \mathrm{Sym}^{r+2} \oplus \det^{-1}$!

**Fact** $\chi_0 = \chi_1^s \alpha^{-1}$, $\alpha : \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \longmapsto ad^{-1}$.

**Known Key Property** $\pi[\mathfrak{m}_{I_1}^3]$ is multiplicity free:

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

$$0 \qquad e_0 \ \chi_0^\vee \qquad e_0' \ (\chi_0^s)^\vee \qquad e_1 \ \chi_1^\vee \qquad e_1' \ (\chi_1^s)^\vee$$

$$gr(\pi(\bar\rho)^\vee) \qquad \chi_0^\vee \alpha \ \overset{y}{\bullet \!\!-\!\!-} \bullet \overset{z}{-\!\!-\!\!\bullet} \ \chi_0^\vee \alpha^{-1} = (\chi_1^s)^\vee$$

$z \cdot e_0 = 0 \Leftrightarrow y \cdot e_0' = 0$

$$U(\mathfrak{g})/J = \mathbb{F}[y,z]/(yz).$$

we get $(\chi_0^\vee \otimes \mathbb{F}[y]) \oplus ((\chi_0^s)^\vee \otimes \mathbb{F}[z]) \oplus (\chi_1^\vee \otimes \mathbb{F}[y]) \oplus (\alpha_1^s \mathbb{1}^\vee \otimes \mathbb{F}[z]) \xrightarrow[\text{Nakayama}]{\text{graded}} gr(\pi(\bar\rho)^\vee) \Rightarrow m_{\mathfrak{p}_0}(gr(\pi(\bar\rho)^\vee)) \le 2$

E.g. $f=2$, $\bar\rho$ irred,

then $\pi(\bar\rho)^{I_1} = 8$-dim'l. So $\mathrm{gr}(\pi^\vee)$ is $\boxed{U(\mathfrak{g})/J}$-mod with 8 generators

$\qquad\qquad \| \qquad \mathbb{F}[y_0, y_1, z_0, z_1]/(y_0 z_0, y_1 z_1)$.

Serre weights

$\sigma_1 = \boxed{(r_0, r_1)}$ — $(r_0+1, p-2-r_0) = \sigma_4'$, $(p-r-r_0, r_1-1)$ — $(p-3-r_0, p-1-r_1)$

$\sigma_2 = (r_0-1, p-2-r_1)$ — $\boxed{(p-1-r_0, p-1-r_1)} = \sigma_1'$ $\leftrightarrow e_1'$, $(r_0-2, r_1)$ — $(p-r_0, r_1-1)$

$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$

$\sigma_3 = (p-1-r_0, p-3-r_1)$ — $(p-r_0, r_1+1) = \sigma_2'$, $(r_0-1, p-4-r_1)$ — $(r_0-2, r_1+2)$

$\sigma_4 = (p-2-r_0, r_1+1)$ — $(r_0, r_1+2) = \sigma_3'$ $\leftarrow \chi_\alpha$, $(p-3-r_0, p-3-r_1)$ — $(r_0+1, p-4-r_1)$

e.g. when $f=2$ & $\bar\rho$ reducible.

$\boxed{\sigma_1 \to \sigma_1' \to \cdots}$ PS $\pi_0$

$\sigma_2 \leftrightarrow \sigma_3$ } supersingular. $\pi_1$

$\boxed{\sigma_4 \to \sigma_4' \to \cdots}$ PS $\pi_2$

Dually, $\chi_{\sigma_1}^\vee \longleftrightarrow e_1$

$y_0$: $\chi_0^\vee \alpha_0$, $y_1$: $\chi^\vee \alpha_1$, $z_0$: $\chi_0^\vee \alpha_0^{-1}$, $z_1$: $\chi^{-1}\alpha_1^{-1} = (\chi_{\sigma_3}^s)^\vee$

$\xRightarrow{\text{multi one}}$ $z_1 \cdot e_1 = 0$, etc. $\xleftrightarrow{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}$ $y_1 \cdot e_1' = 0$

$\Rightarrow \left(\chi_1^\vee \otimes \mathbb{F}[y_0, y_1, z_0]/(y_0 z_0)\right) \oplus \left((\chi_1^s)^\vee \otimes \mathbb{F}[y_0, y_1, z_1]/(y_1)\right) \oplus \cdots$

$\qquad \twoheadrightarrow \mathrm{gr}(\pi(\bar\rho))^\vee$.

$\Rightarrow m_{\mathfrak{p}_0}(\mathrm{gr}\,\pi(\bar\rho)^\vee) \le 4$

## § Results in finite length

$\pi(\bar\rho)$: length as $GL_2(L)$-rep'n.

Recall Expectation: $\pi(\bar\rho)$ is irred. + ss. if $\bar\rho$ is irred.

(generic) $\cdot\ \pi(\bar\rho)$ is of length $f+1$ if $\bar\rho$ is reducible.

$$\simeq \underset{\text{PS}}{\pi_0} \oplus \underbrace{\pi_1 \oplus \cdots \oplus}_{\text{s.s.}} \underset{\text{PS}}{\pi_f} \quad \leftarrow \text{if } \bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

then (supposedly)
$\pi_0 = \mathrm{Ind}_B^{GL_2}(\chi_2 \otimes \chi_1 \omega^{-1})$
$\pi_f = \mathrm{Ind}_B^{GL_2}(\chi_1 \otimes \chi_2 \omega^{-1})$.

<u>Thm</u> (1) $\bar\rho$ irred $\Rightarrow$ Expectation

(2) $\bar\rho$ reducible $\Rightarrow$ Expectation for $f=2$.

{ split : [BHHMS2]
non-split : [Hu-Wang].

## § Self-duality of $\pi(\bar\rho)$

<u>Recall</u> complex smooth rep'n $\pi$

$$\pi^\vee := \mathrm{Hom}_{\mathbb{C}}(\pi, \mathbb{C})^\infty \quad \text{smooth dual.}$$

$$\rho : G_L \to GL_2(\mathbb{C}) \xleftrightarrow{\text{LLC}} \pi$$
$$\rho^\vee \simeq \rho \otimes (\det \rho)^{-1} \longleftrightarrow \pi^\vee$$

<u>But</u> for mod $p$ rep'n:

<u>Problem</u> $\mathrm{Hom}_{\mathbb{F}}(\pi_1, \mathbb{F})^\infty = 0$ (most of the time).

Need a new version of duality ("smooth dual").

$$GL_2(L) \curvearrowright \underset{\text{adm.}}{\pi} \rightsquigarrow \underset{\substack{\text{f.g. over}\\ \Lambda = \mathbb{F}[\![I_1/Z_1]\!]}}{\pi^\vee} \rightsquigarrow \underset{\substack{\text{f.g. } \Lambda\text{-mod}\\ \circlearrowleft\\ GL_2(L)}}{\mathrm{Ext}^i_\Lambda(\pi^\vee, \Lambda)} \rightsquigarrow \underset{\text{sm adm rep'n.}}{(\mathrm{Ext}^i_\Lambda(\pi^\vee, \Lambda))^\vee}$$

Grade: $j_\Lambda(M) := \min\{i \geq 0,\ \mathrm{Ext}^i(M, \Lambda) \neq 0\}$, $M$ = f.g. $\Lambda$-mod.

<u>Remark</u> Have $GK(\pi) + j_\Lambda(\pi^\vee) = \dim \Lambda$ $(= 3f)$.

So $j_\Lambda(M) \nearrow\ \Rightarrow M$ size $\searrow$.

Auslander condition on $\Lambda$: $\forall N \subseteq \mathrm{Ext}^i_\Lambda(M, \Lambda)$, $j_\Lambda(N) \geq i$.

satisfied by $\mathbb{F}[\![I_1/Z_1]\!]$ (cf. Venjakob 02).

(i.e. $i \nearrow\ \Rightarrow \mathrm{Ext}^i_\Lambda(M, \Lambda)$ size $\searrow$).

Def'n $M$ is Cohen-Macaulay if $\exists$ only one $i$, s.t. $\mathrm{Ext}^i_\Lambda(M,\Lambda)\neq 0$.

Def'n $M$ = f.g. $\Lambda$-mod & compatible with $GL_2(L)$-action

Say $M$ is self-dual if $\mathrm{Ext}_\Lambda^{j_\Lambda(M)}(M,\Lambda)\simeq M$,

$M$ is essentially self-dual if $\mathrm{Ext}^{j(M)}(M,\Lambda)\simeq M\otimes$(up to twist)

(determined by central characters).

E.g. $\cdot\ \pi=\mathrm{Ind}_B^G\chi$, PS. ($\dim\Lambda=3f$) $\Rightarrow j_\Lambda(\pi^\vee)=2f$.

Kohlhaase: $\mathrm{Ext}^{2f}(\pi^\vee,\Lambda)=(\mathrm{Ind}_B^{GL_2}\chi^{-1}\cdot\alpha_B)^\vee$.

$\alpha_B=\omega\otimes\omega^{-1}$ modulo char.

Moreover, $\pi^\vee$ is CM.

$\Rightarrow GL_2(\mathbb{Q}_p)$, $\pi(\bar\rho)=(\pi_0-\pi_1)$

So $\pi(\bar\rho)^\vee$ is essentially self-dual.

$\cdot\ \pi$ = ss. for $GL_2(\mathbb{Q}_p)$

$\mathrm{Ext}^{2f}(\pi^\vee,\Lambda)\simeq\pi^\vee\otimes\zeta\cdot\det$

(not complete proof yet. see Paskunas' student's master thesis).

$\cdot$ Complete cohomology $\widetilde H^1$(Shimura curve)

$\widetilde H^0$(Shimura set)

Emerton: $E_2^{i,j}=\mathrm{Ext}^i_\Lambda(\widetilde H_j,\Lambda)\Rightarrow\widetilde H_{d-(i+j)}$.

Thm1 $\pi(\bar\rho)^\vee$ is ess. self-dual

Proof $GK(\pi(\bar\rho))=f\Rightarrow M_\infty$ is flat mod over $R_\infty=\mathcal{O}[\![x_1,\cdots,x_g]\!]$

$S_\infty$

patched module

(assume $\bar\rho$ generic)

and $\pi(\bar\rho)^\vee\simeq M_\infty/\mathfrak{m}_{R_\infty}$.

i.e. $M_\infty$ defines a Koszul complex resolution of $\pi(\bar\rho)^\vee$.

ok, if one knows $M_\infty$ is self-dual.

Solve: $\begin{cases} M_\infty / \mathfrak{m}_{S_\infty} \simeq \widetilde{H}_0 & \text{complete homology} \\ R_\infty / \mathfrak{m}_{S_\infty} \simeq \widetilde{\mathbb{T}}_{\mathfrak{m}} & \text{Hecke algebra; completed intersection ring.} \end{cases}$

Fact $\begin{cases} M \text{ over } A, \ A \text{ complete inter'n} \Rightarrow \text{Gorenstein} \\ M \text{ ess self-dual} \qquad \text{Some self-dual property.} \end{cases}$

$\Rightarrow M/\mathfrak{m}_A M$ is ess self-dual.

## § The semisimple case in $\bar{\rho}$

**Thm 2** $\pi(\bar{\rho})$ is generated by its K-socle $= \bigoplus_{\sigma \in W(\bar{\rho})} \sigma$, as $GL_2(L)$-rep'n.

($\Rightarrow$ generated by $\pi(\bar{\rho})^{K_1}$).

Caution: f.g. $\not\Rightarrow$ finite length.

**Lemma 3** Let $\pi'$ be a subquotient of $\pi(\bar{\rho})$ $\Rightarrow$ $\mathbb{V}_B(\pi')$ is a subquot of $\mathbb{V}_B(\pi(\bar{\rho}))$

(i) $\dim \mathbb{V}_B(\pi') = m_{\mathfrak{p}_0}(\pi')$ where $\dim \mathbb{V}_B(\pi(\bar{\rho})) = 2^f$

(ii) if $\pi'$ is subrep of $\pi(\bar{\rho})$, then $= \text{length } \mathrm{soc}_K(\pi)$

$\dim \mathbb{V}_B(\pi') = \text{length}(\mathrm{soc}_K(\pi'))$ $= m_{\mathfrak{p}_0}(\pi(\bar{\rho}))$

(iii) if $\pi' \neq 0$ is a quotient of $\pi(\bar{\rho})$, then $\mathbb{V}_B(\pi') \neq 0$.

Rmk A priori: don't know length of $\pi(\bar{\rho})$

Can happen that $\exists\ \infty$-many subquotient of $\pi(\bar{\rho})$, $\mathbb{V}_B(-) = 0$.

Lemma 3 $\Rightarrow$ Thm 2

$\pi' :=$ generated by $\mathrm{soc}_K(\pi(\bar{\rho})) \subseteq \pi(\bar{\rho}) \twoheadrightarrow \pi''$.

$\Rightarrow \dim \mathbb{V}_B(\pi') = \text{length } \mathrm{soc}_K(\pi(\bar{\rho})) = \dim \mathbb{V}_B(\pi(\bar{\rho}))$

So $\dim \mathbb{V}_B(\pi'') = 0$

$\overset{(iii)}{\Rightarrow} \pi'' = 0$.

<u>Pf of lem 3</u> (i) easy: $\mathbb{V}_B$, $m_{\beta_0}(-)$ exact

$\Rightarrow \dim \mathbb{V}_B(\pi(\bar{\rho})) = m_{\beta_0}(\pi(\bar{\rho})^\vee)$ ok

$\Rightarrow$ ok for any subquot $\pi'$.

(ii) $(*)$ $\dim \mathbb{V}_B(\pi') \le \text{length}\, \mathrm{soc}_K(\pi')$

To prove $m_{\beta_0}(\pi') \le \text{length}\, \mathrm{soc}_K(\pi')$

Find a graded module $N' \twoheadrightarrow \mathrm{gr}(\pi')$ satisfying

$m_{\beta_0}(N') = \text{length}(\mathrm{soc}_K \pi')$.

(iii) Consider $0 \to \pi'' \to \pi(\bar{\rho}) \to \pi' \to 0$

$0 \to \pi'^\vee \to \pi(\bar{\rho})^\vee \to \pi''^\vee \to 0$

$\mathrm{Ext}^i(-,\Lambda)$ $\rightsquigarrow$ $0 \to \mathrm{Ext}^{2f}(\pi''^\vee,\Lambda) \to \mathrm{Ext}^{2f}(\pi(\bar{\rho})^\vee,\Lambda) \xrightarrow{\gamma} \mathrm{Ext}^{2f}(\pi'^\vee,\Lambda)$

contravariant

$\to \mathrm{Ext}^{2f+1}(\pi''^\vee,\Lambda) \to \mathrm{Ext}^{2f+1}(\pi(\bar{\rho})^\vee,\Lambda) = 0$

CM property

Define $\tilde{\pi}' := (\mathrm{Im}\,\gamma \otimes \zeta^{-1})^\vee$ sm adm rep'n of $GL_2(L)$.

So $\tilde{\pi}' \hookrightarrow \pi(\bar{\rho})$.

<u>Key</u> $m_{\beta_0}(\pi') = m_{\beta_0}(\tilde{\pi}')$ + (ii) $\Rightarrow$ $\mathbb{V}_B(\pi') \neq 0$.

By definition, $\exists$ sequence

$$0 \to \mathrm{Im}\,\gamma \to \underbrace{\mathrm{Ext}^{2f}(\pi'^\vee,\Lambda)}_{j(-)=2f} \to \mathrm{Ext}^{2f+1}(\pi''^\vee,\Lambda) \to 0$$

(Auslander condition $j(-) \ge 2f+1$ $\Rightarrow m_{\beta_0}(-) = 0$).

(<u>Fact</u> $j(\mathrm{Ext}^{j(M)}(M,\Lambda)) = j(M)$).

$\Rightarrow m_{\beta_0}(\mathrm{Im}(\gamma)) = m_{\beta_0}(\mathrm{Ext}^{2f}(\pi'^\vee,\Lambda)) = m_{\beta_0}(\pi'^\vee)$. $\square$

a general fact.

<u>Rmk</u> Have proved $\mathbb{V}_B(\pi') \neq 0 \iff \mathbb{V}_B(\tilde{\pi}') \neq 0$

i.e. $m_{\beta_0}(\pi') \neq 0 \iff m_{\beta_0}(\tilde{\pi}') \neq 0$ (only this).

Proof of Thm 1 (i) $\bar\rho$ irred $\Rightarrow$ $\pi(\bar\rho)$ irred

(Breuil-Paškūnas) MAMS: $\pi(\bar\rho)$ generated by $\mathrm{soc}_K(\pi(\bar\rho))$ & $\bar\rho$ irred
$\Rightarrow$ $\pi(\bar\rho)$ irred.

Recall Irreducibility creterion:

if $\forall \sigma \in \mathrm{soc}_K(\pi(\bar\rho))$, $\sigma$ generates $\pi(\bar\rho)$,
then $\pi(\bar\rho)$ is irred.

(Used a fact: $\forall\, 0 \neq \pi' \subseteq \pi(\bar\rho)$, $\pi' \cap \mathrm{soc}_K(\pi(\bar\rho)) \neq 0$)

It suffices to prove: $\forall \sigma \hookrightarrow \pi(\bar\rho)$, $\sigma$ generates $\mathrm{soc}_K \pi(\bar\rho)$.

Rmk $f=3$, $\bar\rho$ irred, a bit more complicated (but still valid).

When $\bar\rho$ reducible split ($f=2$):

Conj $\pi(\bar\rho) = \underset{\substack{\uparrow \\ PS}}{\pi_0} \oplus \underbrace{\cdots}_{s.s.} \oplus \underset{\substack{\uparrow \\ PS}}{\pi_f}$

Easy to prove: $\pi_0 \hookrightarrow \pi(\bar\rho) \hookleftarrow \pi_f$.

Claim: $\pi(\bar\rho) = \pi_0 \oplus \pi_f \oplus \underset{\substack{\| \\ \pi'}}{(sth)}$ (need self-duality).

$\rightsquigarrow$ $\pi_0 \oplus \pi_f \hookrightarrow \pi(\bar\rho) \twoheadrightarrow \pi_0 \oplus \pi_f \Rightarrow$ isom.

- $f=2 \Rightarrow \pi'$ is irred and supersingular.

Why $f=3$ fails to be valid?

$$\left.\begin{aligned} \pi_0 &: \boxed{\sigma_1} \\ \pi_1 &: \boxed{\sigma_2 \leftrightarrow \sigma_3 \hookleftarrow \sigma_4} \\ \pi_2 &: \boxed{\sigma_5 \leftrightarrow \sigma_6 \hookleftarrow \sigma_7} \\ \pi_3 &: \boxed{\sigma_8} \end{aligned}\right\} \hookrightarrow \pi(\bar\rho).$$

## § Reducible non-split case

<u>Thm 4</u> $\bar\rho = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, $* \neq 0$. Assume $f=2$.

Then $\pi(\bar\rho)$ has length 3 : $\begin{matrix} \pi_0 & - & \pi_1 & - & \pi_2 \\ \text{PS} & & \text{s.s.} & & \text{PS} \end{matrix}$

<u>Thm 5</u> $\pi(\bar\rho)$ is generated by $D_0(\bar\rho) = \pi(\bar\rho)^{K_1}$ as a $GL_2(L)$-rep'n.

<u>Proof idea of Thm 5</u>

- $\pi_0 \hookrightarrow \pi(\bar\rho)$ : because $\pi_0 = \text{PS}$ (use Serre wt + Hecke action).
- self-duality of $\pi(\bar\rho) \Rightarrow \pi(\bar\rho) \twoheadrightarrow \pi_2$

  Moreover, $\pi_0 = \operatorname{soc}_{GL_2(L)} \pi(\bar\rho)$ b/c $\bar\rho$ non-split

  $\Rightarrow \pi_2 = \operatorname{cosoc}_{GL_2(L)}(\pi(\bar\rho))$.
- <u>Lemma</u> Let $\tau \subseteq \pi(\bar\rho)|_K$, if for some $\sigma$ (irred rep'n of $K$), some $i$,

$$\begin{tikzcd} & \operatorname{Ext}^i_K(\sigma,\tau) \arrow[dl, dashed] \arrow[d] \arrow[dr, dotted, "\beta"] & \\ \operatorname{Ext}^i_K(\sigma,\pi') \arrow[r] & \operatorname{Ext}^i_K(\sigma,\pi(\bar\rho)) \arrow[r] & \operatorname{Ext}^i_K(\sigma,\pi_2) \end{tikzcd}$$

composite is nonzero

Then $\pi(\bar\rho)$ is generated by $\tau$ as $GL_2(L)$-rep'n.

Pf Let $\pi' = \langle GL_2(L).\tau \rangle \subseteq \pi(\bar\rho) \twoheadrightarrow \pi_2$

then $\pi' \neq \pi(\bar\rho)$ iff the composite is $0$. } cosocle

Assume $\pi' \neq \pi(\bar\rho)$, then the composite $\beta = 0 \rightsquigarrow$ contradiction

Choice of $i$ in Lemma: $\boxed{i = 2f}$. □

<u>Thm 5 ⇒ Thm 4</u> (when $f=2$).

<u>Step 1</u> Known: $\pi_0 \hookrightarrow \pi(\bar\rho)$, study $\pi(\bar\rho)/\pi_0$ : what is its $GL_2(L)$-socle ?

<u>Fact</u> ($f \geq 2$) If $\pi'$ irred rep'n of $GL_2(L)$, $\pi'$ non-s.s.

Assume $\operatorname{Ext}^1_{GL_2}(\pi', \pi_0) \neq 0$. Then $\pi' \cong \pi_0$.

$\pi(\bar{\rho}) = (\pi_0 - \pi')$. Here $\pi' \overset{\text{irred}}{\hookrightarrow} \pi(\bar\rho)/\pi_0$.

Fact $\Rightarrow$ either $\pi'$ s.s. or $\pi' = \pi_0$.

Step 2 $\pi'$ must be s.s.

Use ordinary part of $\pi(\bar\rho)$:

Recall Emerton defines $\mathrm{ord}_B : G\text{-rep} \to \overset{T}{\overset{\|}{\text{Torus}}}\text{-rep}$

$\rightsquigarrow \mathrm{Hom}(\mathrm{Ind}_{\bar B}^{G} U, V) \simeq \mathrm{Hom}_T(U, \mathrm{ord}_B V)$.

$\Rightarrow \mathrm{ord}_B(\mathrm{Ind}_{\bar B} U) = U$.

On the other hand $\mathrm{ord}_B(\text{s.s.}) = 0$.

Key If $0 \to \pi_0 \to \mathcal{E} \to \pi_0 \to 0$ self-ext'n.

then $0 \to \underset{\substack{\updownarrow \\ \chi\text{-char}}}{\mathrm{ord}_B(\pi_0)} \to \mathrm{ord}_B(\mathcal{E}) \to \underset{\substack{\| \\ \chi}}{\mathrm{ord}_B(\pi_0)} \to 0$ exact

Thm $\mathrm{ord}_B(\pi(\bar\rho))$ is semi-simple (Hu-Breuil-Ding)

$\Rightarrow \pi'$ is a supersingular rep'n;

$(\pi_0 - \pi') \hookrightarrow \pi(\bar\rho)$.

Step 3 Study $\pi(\bar\rho)/(\pi_0 - \pi')$ (?)

Expectation: it is $\pi_2$ (PS).

Apply Thm 5 $\Rightarrow \pi(\bar\rho)$ is generated by $D_0(\bar\rho)$.

So does $\pi(\bar\rho)/(\pi_0-\pi')$: generated by $D_0(\bar\rho)/(D_0(\bar\rho)\cap(\pi_0 - \pi'))$

Computation: this quotient is generated by $\sigma_4$.

Frob reciprocity $\Rightarrow$ $c\text{-}\mathrm{Ind}_{GL_2(\mathcal{O}_L)\cdot L^\times}^{GL_2(L)} \sigma_4 \twoheadrightarrow \underset{\substack{\| \\ Q}}{\pi(\bar\rho)/(\pi_0 - \pi')}$

satisfy $\mathrm{cosoc}_G Q = \pi_2$, PS.

($f=2$: $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \longleftrightarrow \mathrm{soc}_K \pi_2$).

Lemma If $Q$ = quotient of $c\text{-}\mathrm{Ind}\, \sigma_4$, s.t.

$\mathrm{cosoc}_G Q \simeq PS$, $\pi_2 \simeq$ c-Ind $\sigma_4/(T-\lambda)$, $\lambda \neq 0$.

by Barthel-Livné.

Then $Q \simeq$ c-Ind $\sigma_4/(T-\lambda)^n$ for some $n$.

i.e. $Q \simeq (\underbrace{\pi_2 - \pi_2 - \cdots - \pi_2}_{n \text{ copies}})$.

$\rightsquigarrow$ Left to prove $n=1$:

self-duality:

$$\exists\, (\underbrace{\pi_0 - \cdots - \pi_0}_{n \text{ copies}}) \hookrightarrow \pi(\bar{\rho})$$

$\Rightarrow$ So $n=1$.

$\Rightarrow$ $\pi(\bar{\rho}) = (\pi_0 - \pi_1 - \pi_2')$. $\square$