On the Archimedean Arithmetic Smooth Matching

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Period integrals

- G - reductive group over $\mathbb{Q}$; $[G] = G(\mathbb{Q})\backslash G(\mathbb{A})$.
- H - subgroup of G.
- We may consider period integrals over H for automorphic forms φ on G

$$P(\varphi) = \int_{[H]} \varphi(h)dh.$$

- Period integrals are closely related to L-values.

Waldspurger formula

- B - quaternion algebra over $\mathbb{Q}$ containing a quadratic field E.
- $(G,H) = (\mathbb{Q}^\times \backslash B^\times, \mathbb{Q}^\times \backslash E^\times)$.
- Let π be a cuspidal automorphic representation on $B_{\mathbb{A}}^\times$ with $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$.

Waldspurger formula

$$|P(\varphi)|^2 \doteq L(1/2, \pi_E), \quad \varphi \in \pi.$$

Cycles on Shimura varieties

- We may also consider the Shimura varieties Y and X associated to the groups H and G.
- Usually, the embedding $H \hookrightarrow G$ induces a finite morphism $Y \to X$ so that we may view Y as a cycle on X.
- The height pairing for such cycles in X are closely related to L-derivatives.

The Gross-Zagier formula

- $G = \mathrm{GL}_2/\mathbb{Q}$.
- $H = E^\times$ with E an imaginary quadratic field.
- The pair (G, H) gives the Heegner points on modular curves.
- $\varphi \in S_2(\Gamma_0(N))^{\mathrm{new}}$.
- Assume the Heegner hypothesis: any prime factor of N is split in E.
- $X = X_0(N)$ - the modular curve over $\mathbb{Q}$ with level $\Gamma_0(N)$.
- The complex points of $X_0(N)$ parametrizes isogenies of elliptic curves over $\mathbb{C}$ with kernel $\mathbb{Z}/N\mathbb{Z}$.
- $z = (\mathbb{C}/\mathcal{O}_E \to \mathbb{C}/\mathcal{N}^{-1}) \in X(\mathbb{C})$ with $\mathcal{O}_E/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$.
- $z \in X(H_E)$.
- $Y^0 = \sum_{\sigma \in \mathrm{Gal}(H_E/E)} (z - [\infty])^\sigma \in J(E)$ with J the Jacobian of X.
- $P(\varphi)$ - the φ-component of Y^0 in $J(E)_\mathbb{C}$.

Gross-Zagier formula

We have the following identity

$$\langle P(\varphi), P(\varphi)\rangle_{\mathrm{NT}} \doteq L'(1, \varphi_E).$$

where $\langle \cdot, \cdot \rangle_{\mathrm{NT}}$ is the Neron-Tate height on $J(\bar{\mathbb{Q}})_\mathbb{C}$.

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- Next, via the Gross-Zagier case, we introduce the RTF approach, especially the archimedean arithmetic smooth matching.

The automorphic kernel

- In general, consider G a reductive group over $\mathbb{Q}$ with Z its center.
- $G(\mathbb{A})$ acts on $L^2([Z\backslash G])$ by right multiplication, denoted by R.
- The space of Schwartz functions $\mathcal{S}(G(\mathbb{A}))$ acts on $L^2([Z\backslash G])$

$$R(f)\varphi = \int_{G(\mathbb{A})} f(g)R(g)\varphi dg, \quad f \in \mathcal{S}(G(\mathbb{A})).$$

It is an integral operator with kernel

$$K_f(x,y) = \int_{[Z]} \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma zy) d^\times z.$$

- Usually, a *relative trace* is a distribution on $\mathcal{S}(G(\mathbb{A}))$ given by integration of K_f over two specified subgps.

The L-value side

- $G' = \mathrm{GL}_{2,E}$.
- $\mathrm{GL}_{1,E} \hookrightarrow G'$ by $a \mapsto \begin{pmatrix} a & \\ & 1 \end{pmatrix}$.
- $\mathrm{GU}_w = \{g \in G' | g w^t \bar{g} = \kappa(g) w\}$- similitude unitary group of $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- For Π a cuspidal automorphic representation on $G'(\mathbb{A})$
- $$\mu_s : \varphi \in \Pi \mapsto \int_{[\mathrm{GL}_{1,E}]} \varphi \left[\begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] |a|_E^s d^\times a.$$
  Then $L(1/2 + s, \Pi)$ is nonzero iff $\mu_s \neq 0$ on Π.
- $$\ell : \varphi \in \Pi \mapsto \int_{[Z' \backslash \mathrm{GU}_w]} \varphi(h) \eta(\kappa(h)) dh, \quad \eta = \eta_{E/\mathbb{Q}}$$
  Then Π is the base change of a cusp auto rep on $\mathrm{PGL}_2(\mathbb{A})$ iff $\ell \neq 0$ on Π.

RT for L-value

$$I(s, f') = \int^{\mathrm{reg}}_{[\mathrm{GL}_{1,E} \times Z' \backslash \mathrm{GU}_w]} K_{f'} \left[\begin{pmatrix} a & \\ & 1 \end{pmatrix}, h \right] |a|_E^s \eta(\kappa(h)) d^\times a dh, \quad f' \in \mathcal{S}' = \mathcal{S}(G'(\mathbb{A})).$$

The RT for L-value - geometric side

- For the geometric side, expand $I(s,\cdot)$ into orbital integrals for the action of $\mathrm{GL}_{1,E} \times \mathrm{GU}_w$ on G'.
- $$E^\times \backslash G'(\mathbb{Q})/\mathrm{GU}_w(\mathbb{Q}) \hookrightarrow T_1(\mathbb{Q})\backslash S(\mathbb{Q}), \quad [g] \mapsto [g \circ w], \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  where S is the G'-variety of nondeg 2×2 Hermitian matrices w.r.t $E/\mathbb{Q}$ and $T_1 = \mathrm{GL}_{1,E} \times \mathrm{GL}_1$.
- $f' \rightsquigarrow \Phi_{f'} \in \mathcal{S}(S(\mathbb{A}))$ via the map $G'/\mathrm{U}_w \to S$
- The regular semisimple orbits $T_1(\mathbb{Q})\backslash S(\mathbb{Q})^{\mathrm{reg}} \cong \mathbb{Q}^\times \setminus \{1\}$.

Geometric expansion of $I(s,\cdot)$

$$I(s,f') = \sum_x \mathcal{O}(s,x,\Phi) + I_{\mathrm{sing}}(s,\Phi).$$

For each x with $\Phi = \otimes_v \Phi_v$,

$$\mathcal{O}(s,x,\Phi) = \prod_v \mathcal{O}(s,x,\Phi_v), \quad \mathcal{O}(s,x,\Phi_v) = \int_{T_1(\mathbb{Q}_v)} \Phi_v(t \circ s(x))\xi_{s,v}(t)dt, \quad \xi_{s,v}(a,z) = |a|_{E,v}^{-s}\eta_v(z).$$

The RT for L-value - the spectral side

- For the spectral side, expand $I(s,\cdot)$ into Bessel distributions (Hecke eigen invariant distributions).
- Spectral decomposition of the $\mathcal{S}'$-mod $L^2([Z'\backslash G'])$

$$L^2([Z'\backslash G']) = L^2_{\text{cusp}}([Z'\backslash G']) \bigoplus L^2_{\text{Eis}}([Z'\backslash G']), \quad L^2_{\text{cusp}}([Z'\backslash G']) = \widehat{\bigoplus_{\Pi}} L^2_{\Pi}.$$

- $K_{f'} = K_{f',\text{cusp}} + K_{f',\text{Eis}}, \quad K_{f',\text{cusp}} = \sum_{\Pi} K_{f',\Pi}.$

Spectral expansion of $I(s,f')$

$$I(s,f') = \sum_{\Pi} I_{\Pi}(s,f') + I_{\text{Eis}}(s,f')$$

where for each Π, the Bessel distribution

$$I_{\Pi}(s,f') = \mathcal{B}_{\Pi}^{\mu_s,\ell}(f') = \sum_{\varphi\in \text{OB}(\Pi)} \mu_s(\Pi(f')\varphi)\overline{\ell(\varphi)}.$$

The RT for toric period

- For the toric period side, we consider a distribution on $B^\times$ with respect to the subgroup $E^\times$.

The RT for toric period

$$J(f) = \iint_{[\mathbb{Q}^\times \backslash E^\times]^2} K_f(t_1, t_2) dt_1 dt_2, \quad f \in \mathcal{S}(B_{\mathbb{A}}^\times)$$

- It has the geometric expansion

$$J(f) = \sum_x \mathcal{O}(x, f) + J_{\text{sing}}(f)$$

where x denotes the regular semisimple orbits $E^\times \backslash (B^\times)^{\text{reg}} / E^\times \hookrightarrow \mathbb{Q}^\times \setminus \{1\}$.

- It has the spectral expansion

$$J(f) = \sum_\pi J_\pi(f) + J_{\text{Eis}}(f)$$

where for each π, the Bessel distribution

$$J_\pi(f) = \mathcal{B}_\pi^{P,P}(f) = \sum_{\varphi \in \text{OB}(\pi)} P(\pi(f)\varphi)\overline{P(\varphi)}.$$

Comparison

We shall compare $I(\Phi) = I(0, \Phi)$ and $J(f)$ for (Φ, f) at the geometric sides. As orbital integrals are Eulerian, the comparison reduces to local.

- Matching of orbits: for separable quadratic extension E/F

$$T_1(F)\backslash S(F)^{\mathrm{reg}} \xrightarrow{\sim} F^\times \setminus \{1\} \xleftarrow{\sim} \bigsqcup_{E\subset B} E^\times\backslash(B^\times)^{\mathrm{reg}}/E^\times.$$

- Let F be a local field. Let $\Phi \in \mathcal{S}(S(F))$ and $(f_B \in \mathcal{S}(B^\times))_B$. We call Φ matches with $(f_B)_B$ if for any $x \in F^\times \setminus \{1\}$

$$\mathcal{O}(x, f_B) = \mathcal{O}(x, \Phi)$$

  when x is in the image of inv_B.

Local comparison

- (Existence of smooth matching). Given Φ, there exists $(f_B)_B$ matches with Φ. Conversely, given any $(f_B)_B$, there exists some Φ matches with $(f_B)_B$.
- (Fundamental Lemma). Consider the unramified situation. Then

$$\Phi_{f'} \sim (b(f'), 0), \quad f' \in \mathcal{H}'.$$

  Here, b is the base change morphism from the unramified Hecke algebra $\mathcal{H}'$ of $G'(F)$ to that of $\mathrm{GL}_2(F)$.

The proof of the Waldspurger formula

Global comparison

Let $f = \otimes_v f_v \in \mathcal{S}(B_{\mathbb{A}}^{\times})$. Let $f' = \otimes_v f_v' \in \mathcal{S}'$ be purely matching with f, that is, for each v,

$$\Phi_{f_v'} \sim (f_v', 0).$$

Then

$$I(f') = J(f).$$

By the principal of independence of characters, for any π on $B_{\mathbb{A}}^{\times}$ with $\mathrm{Hom}_{E_{\mathbb{A}}^{\times}}(\pi, \mathbb{C}) \neq 0$

$$I_{\pi_E}(f') = J_{\pi}(f).$$

This gives the Waldspurger formula.

The RT for height

- $G = \mathrm{GL}_2/\mathbb{Q}$.
- U - an open compact subgroup of $G(\mathbb{A}_f)$.
- X - the modular curve with level U. The complex points of X forms a Riemann surface

$$X(\mathbb{C}) \cong G(\mathbb{Q})\backslash\mathcal{H}^{\pm} \times G(\mathbb{A}_f)/U \cup \{\text{cusps}\}.$$

- $\mathcal{S} = \mathcal{S}(U\backslash G(\mathbb{A}_f)/U)$ bi-U-invariant Schwartz functions.
- Hecke action on the Jacobian J of X

$$R : \mathcal{S} \longrightarrow \mathrm{End}(J)_{\mathbb{C}}.$$

- E - imaginary quadratic field with a fixed embedding $E^{\times} \hookrightarrow G$;
- z_0 - the unique point in $\mathcal{H}$ fixed by $E^{\times}$.
- For any $g \in G(\mathbb{A}_f)$, consider the CM points

$$[g] = [z_0, g] \in X(\bar{\mathbb{Q}}), \quad [g]^0 = [g] - [\infty]_g \in J(\bar{\mathbb{Q}}).$$

- Heegner cycle

$$P^0 = \int_{E^{\times}\backslash\widehat{E}^{\times}/U\cap\widehat{E}^{\times}} [t]^0 dt \in J(E).$$

The RT for height of the Heegner cycle

$$H(f) = \langle R(f)P^0, P^0\rangle_{\mathrm{NT}}, \quad f \in \mathcal{S}.$$

The global comparison

Global comparison (Tian-Yuan-Zhang-C)

- S - finite set of nonarchimedean ramified places.
- $f_S \in \mathcal{S}_S^0$ - "nice" test functions to simplify the computation of height.
- $f'_S \in \mathcal{S}'_S$ - purely matching with f_S.
- $f'_\infty \in \mathcal{S}_\infty$ - Gaussian.

Then the following two distributions on the spherical Hecke algebra $\mathcal{H}'^{(S)}$

$$H\left(f_S \otimes b\left(f'^{(S)}\right)\right), \quad I'\left(f'_\infty \otimes f'_S \otimes f'^{(S)}\right), \quad f'^{(S)} \in \mathcal{H}'^{(S)}$$

are equal up to a coherent distribution.

The global comparison

- a linear functional ℓ' on $\mathcal{H}'^{(S)}$ is called *coherent* if

$$\ell' = \sum_{\Pi} \ell'_{\Pi} + \text{Eisenstein part}.$$

where $\Pi = \pi_E$ is the base change lifting of a cuspidal automorphic representation π on $\mathrm{PGL}_2(\mathbb{A})$ with the root number of $\varepsilon(1/2, \Pi) = +1$.

- The most important example of coherent functionals comes from the distribution J.

For any π on $G(\mathbb{A})$ discrete of weight two with $\varepsilon(1/2, \pi_v) = +1$ for any nonarchimedean v

$$I'_{\pi_E}(f') = H_{\pi}(f)$$

for any purely matching f' and f. This gives the Gross-Zagier formula of Yuan-Zhang-Zhang.

Semi-global comparison

- The Neron-Tate height pairing is a sum of local heights so that the distribution $H(f)$ is also a sum of distributions $H_v(f)$ for local heights

$$H_v(f) = \langle R(f)P, P\rangle_v, \quad f \in \mathcal{S}_S^0 \otimes \mathcal{H}^{(S)}$$

where the local height $\langle \cdot, \cdot \rangle_v$ is a pairing on divisors of X_v with disjoint supports.

- The distribution I' is also a sum of distributions indexed by places,

$$I'_v(f') = \sum_x \mathcal{O}'(x, f'_v)\mathcal{O}\left(x, f'^{(v)}\right).$$

Here, $\mathcal{O}'(x, f'_v)$ is the derivative of $\mathcal{O}(s, x, f'_v)$ at $s = 0$.

Semi-global comparison

For each v, (with test functions as in the global comparison)

$$H_v\left(f_S \otimes b\left(f'^{(S)}\right)\right), \quad I'_v\left(f'_\infty \otimes f'_S \otimes f'^{(S)}\right), \quad f'^{(S)} \in \mathcal{H}'^{(S)}$$

are equal up to a *coherent* distribution.

- In the following, we focus on the archimedean theory.

Archimedean arithmetic smooth matching

Step 1. Reduction from the semi-global comparison to the local one.

There exists $h_\infty \in C^\infty(\mathbb{C}^\times \backslash G(\mathbb{R}) - \mathbb{C}^\times)$ *with compact support in* $\mathbb{C}^\times \backslash G(\mathbb{R})$ *such that for any* $f_S \in \mathcal{S}_S^0$,

$$H_\infty\left(f_S \otimes f^{(S)}\right) = \sum_x \mathcal{O}\left(x, h_\infty \otimes f_S \otimes f^{(S)}\right), \quad f^{(S)} \in \mathcal{H}^{(S)}$$

up to coherent distributions on $\mathcal{H}^{(S)}$.

Step 2. Local comparison.

There exists $h'_\infty \in \mathcal{S}(G(\mathbb{R}))$ *such that for any* $x \in \mathbb{R}^\times \setminus \{1\}$

$$\mathcal{O}'(x, f'_\infty) - \mathcal{O}(x, h_\infty) = \mathcal{O}(x, h'_\infty).$$

A general problem

- In fact, Step 1 can be formulated in a very general setting.
- (G, H_1, H_2) - a triple of reductive groups over $\mathbb{Q}$ where H_1, H_2 are subgroups of G.
- (X, Y_1, Y_2) - the triple of Shimura varieties associated to (G, H_1, H_2). Assume that

$$\dim X + 1 = \operatorname{codim}(Y_1, X) + \operatorname{codim}(Y_2, X).$$

- For each open compact subgroup U of $G(\mathbb{A}_f)$, $(n = \dim X)$

$$\langle \cdot, \cdot \rangle_{U,\infty} : (Z^i(X_U) \times Z^{n+1-i}(X_U))^0 \longrightarrow \mathbb{C}, \quad (Z_1, Z_2) \mapsto \int_{Z_2} g_{Z_1}$$

where $g_{Z_1} \in D^{i-1,i-1}(X_U)$ is the unique harmonic Green current on X_U for Z_1 normalized by

$$\int_{X_U} g_{Z_1} h = 0, \quad h \in \mathcal{H}^{n+1-i,n+1-i}(X_U).$$

The "relative trace" for archimedean local height

$$H_\infty(\phi) = \langle R(\phi)Y_1, Y_2\rangle_\infty \overset{\phi=\phi_f}{=} \mathrm{Vol}(U)\langle R_U(f)Y_{1,U}, Y_{2,U}\rangle_{U,\infty}, \quad \phi \in \mathcal{S}^0.$$

Here,

- For each U, denote by R_U the Hecke correspondence of $C_c^\infty(U\backslash G(\mathbb{A}_f)/U)$ on X_U.
- $\mathcal{S} = C_c^\infty(H_1(\mathbb{A}_f)\backslash G(\mathbb{A}_f))$.
- S - a finite set of places containing ramified places of (G, H_1, H_2).
- $\mathcal{S}^0 = \mathcal{S}_S^0 \otimes \mathcal{S}^{(S)} \subset \mathcal{S}$ - for any $\phi \in \mathcal{S}^0$, the two cycles $R(\phi)Y_1$ and Y_2 are *disjoint supported*.
- Such pairing is independence on the choice of f and the level U of f.

Question: geometric expansion of H_∞?

Note that

$$H_\infty \in \mathrm{Hom}_{H_2(\mathbb{A}^{(S)})}(\mathcal{S}^0, \mathbb{C}).$$

Can we expand H_∞ in terms of orbital integrals?

- To simplify our situation, in the definition of $\mathcal{S}_S^0$, assume there is a place $v_0 \in S$, such that the support of ϕ_v is contained in the elliptic semisimple locus.
- We expect H_∞ admits a geometric expansion in the following sense.

Decomposability

The distribution H_∞ is called *decomposable* provided that there exists

$$h_\infty \in C_c^\infty(H_1(\mathbb{R})\backslash(G(\mathbb{R}) - H_1(\mathbb{R})U_\infty))$$

with compact support on $H_1(\mathbb{R})\backslash G(\mathbb{R})$ such that

$$H_\infty(\phi) = \sum_{\gamma \in H_1(\mathbb{Q})\backslash G(\mathbb{Q})/H_2(\mathbb{Q})} \mathrm{Vol}([H_{2,\gamma}])\mathcal{O}(\gamma, h_\infty \otimes \phi), \quad \phi \in \mathcal{S}^0$$

up to a (H_1, H_2)-coherent distribution.

- We require h_∞ has compact support for the local comparison in the RTF.

- A distribution ℓ on $\mathcal{S}^0$ is called (H_1, H_2)-*coherent* if for any $\phi_S \in \mathcal{S}_S^0$, the following distribution on the spherical Hecke algebra $\mathcal{H}_G^{(S)}$

$$\ell\left(\phi_S \otimes \phi_f\right), \quad f \in \mathcal{H}_G^{(S)}$$

  is
  - *separable*, that is

$$\ell\left(\phi_S \otimes \phi_f\right) = \sum_{\pi} \ell_\pi \mathrm{Tr}_{\pi^{(S)}}(f) + \text{Eisenstein part}$$

    where π runs over cuspidal automorphic representations on $G(\mathbb{A})$
  - (H_1, H_2)-*distinguished*, that is, $\ell_\pi = 0$ unless π is nearly equivalent to a H_1 and H_2-distinguished representations.
- Important example of (H_1, H_2)-coherent distributions is the relative trace I on $H_1 \backslash G$ for H_2: we have the *spectral expansion*

$$I(\phi) = \sum_{\pi} \sum_{\varphi \in \mathrm{OB}(\pi)} P_{H_1}^{\phi}(\overline{\varphi}) P_{H_2}(\varphi) + \text{Eisenstein part}$$

  where

$$P_{H_1}^{\phi}(\varphi) = \int_{H_1(\mathbb{A}) \backslash G(\mathbb{A})} \phi(g) P_{H_1}(\pi(g)\varphi) dg.$$

A rank one case

We consider the following case $(G(\mathbb{R}), H_1(\mathbb{R}))$ of rank 1 and Y_1 a divisor of X:

- $G = \mathrm{U}(V)$. Here, $(V, (\cdot,\cdot))$ is a Hermitian space with respect to an imaginary quadratic field $E/\mathbb{Q}$ of dimension n with signature $(n-1, 1)$ in the archimedean place.
- $H_1 = \mathrm{Stab}_G(Eu)$ with $u \in V$ of norm one.
- H_2 - an anisotropic torus in G.

Theorem (Tian-C)

The distribution H_∞ for the above (G, H_1, H_2) is decomposable if $n \neq 3$.

- For the case $n = 3$, if we moreover require $\mathcal{S}_S^0$ satisfying
  - there is a place v_1 split in E (so that $(G_{v_1}, H_{v_1}) = (\mathrm{GL}_3, \mathrm{GL}_2 \times \mathrm{GL}_1)$) such that ϕ_{v_1} is RSC in the sense that

    $$\phi_{v_1}(g) = \ell(\sigma(g)\varphi)$$

    where σ is a *relatively supercuspidal representation* on G_{v_1}, ℓ is a nonzero functional in $\mathrm{Hom}_{H_{v_1}}(\sigma, \mathbb{C})$ and φ is a nonzero vector of σ.

  then H_∞ is also decomposable.
- There is no distinguished supercuspidal representations for $(\mathrm{GL}_3, \mathrm{GL}_2 \times \mathrm{GL}_1)$.

Local theory of Green functions - the secondary spherical functions

- The key ingredient for the proof is the Green current of Oda-Tsuzuki on X for Y_1.
- We refine their work to fit into the framework of the RTF approach.

The secondary spherical functions

There is the polar decomposition

$$G(\mathbb{R}) = H_1(\mathbb{R}) \exp(\mathfrak{a}) U_\infty, \quad \mathbb{R} \cong \mathfrak{a} \subset \mathfrak{g}.$$

For any s with $\mathrm{Re}(s) > n$, there exists a unique family of secondary spherical functions

$$\phi_s^{(2)} \in C^\infty(H_1(\mathbb{R}) \backslash G(\mathbb{R}) - H_1(\mathbb{R}) U_\infty / U_\infty)$$

characterized by

- the family $\phi_s^{(2)}$ is holomorphic.
- $\phi_s^{(2)} * \Omega = (s^2 - n^2) \phi_s^{(2)}$ where Ω is the Casimir operator on $G(\mathbb{R})$.
- $\phi_s^{(2)}(a_t) - \log t$ is bounded for $t \to 0+$.
- It has the following "large-time behaviour"

$$\phi_s^{(2)}(a_t) = O\left(e^{-(\mathrm{Re} s + n)t}\right), \quad t \to +\infty.$$

Global theory of Green functions - spectral method

The Green function (Oda-Tsuzuki)

Consider the Poincare series

$$G_s(x) = \sum_{\gamma \in H_1(\mathbb{Q})\backslash G(\mathbb{Q})} (\phi_s^{(2)} \otimes 1_U)(\gamma x).$$

For $\mathrm{Re}(s) \gg 0$, the above sum is convergent and $G_s \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathcal{U})$ with $\mathcal{U} = U_\infty \cdot U$.
For any s_1, s_2, the difference $(G_{s_1} - G_{s_2})(g)$ equals the sum of (cusp) + (cont) + (res):

$$\sum_{\pi \in \Pi_{\mathrm{cusp}}(G)^\sigma} \left[\frac{1}{n^2 - s_1^2 - \lambda(\pi)} - \frac{1}{n^2 - s_2^2 - \lambda(\pi)}\right] \sum_{\varphi \in \mathrm{OB}\left(\pi^{\mathcal{U}}\right)} \overline{P_{H_1}(\varphi)}\varphi(g) \tag{cusp}$$

- $\Pi_{\mathrm{cusp}}(G)^\sigma$ - cuspidal unitary automorphic representations π on $G(\mathbb{A})$ which are H-distinguished, that is, $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$.

$$\sum_{\pi \in \Pi_{\mathrm{cusp}}(M)^\sigma/\mathrm{Im}X_M} \sum_{\varphi \in \mathrm{OB}\left(V(\pi)^{\mathcal{U}}\right)} \int_{\mathrm{Im}X_M} \left[\frac{1}{n^2 - s_1^2 - \lambda\left(I_P^G(\xi \circ \pi)\right)} - \frac{1}{n^2 - s_2^2 - \lambda\left(I_P^G(\xi \circ \pi)\right)}\right] \overline{P_{H_1}(E(\varphi, \xi))}E(\varphi, \xi)(g \tag{cont}$$

- $P = MN$ - a minimal parabolic, $M = E^\times \times G_1$ with G_1 a compact unitary group;
- $\Pi_{\mathrm{cusp}}(M)^\sigma$ - cuspidal unitary automorphic representations $\pi = \chi \boxtimes \tau$ on $M(\mathbb{A})$ such that χ factors through $\mathrm{N}_{E/\mathbb{Q}}$ and $\tau = 1$.

$$\left[\frac{1}{n^2 - s_1^2} - \frac{1}{n^2 - s_2^2}\right] \frac{\mathrm{Vol}([H_1])}{\mathrm{Vol}([G])}. \tag{res}$$

Global theory of Green functions

The Green function (Oda-Tsuzuki)

The above spectral expansion of $G_{s_1} - G_{s_2}$ gives the meromorphic continuation of G_s to the whole s-plane which has a simple pole at $s = n$. Moreover,

$$G = \lim_{s \to n} \left(G_s - \frac{1}{n^2 - s^2} \frac{\mathrm{Vol}([H_1])}{\mathrm{Vol}([G])} \right)$$

is a harmonic Green function for Y.

- Oda-Tsuzuki obtains the spectral expansion of G_s and its meromorphic continuation in the L^2-sense.
- As we shall consider its H_2-period integral, the above smooth version of spectral expansion is needed.
- For this, we have the following lemma.

A convergence lemma

Let Φ be a smooth function on $G(\mathbb{A})$ which also belongs to $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathcal{U})$. Assume that

- for each $\pi \in \Pi_{\text{cusp}}(M)^\sigma$, $\varphi \in V(\pi)^{\mathcal{U}}$ and $\xi \in \text{Im}X_M$, $\langle \Phi, E(\varphi, \xi)\rangle_{\text{Pet}}$ is absolutely convergent;
- the function Φ has the following spectral expansion in the L^2-sense

$$\begin{aligned}\Phi(g) = & \sum_{\pi\in\Pi_{\text{cusp}}(G)^\sigma} \sum_{\varphi\in\text{OB}(\pi^{\mathcal{U}})} \langle \Phi, \varphi\rangle_{\text{Pet}}\varphi(g) + \langle \Phi, 1\rangle_{\text{Pet}} \frac{\text{Vol}([H_1])}{\text{Vol}([G])} \\ & + \sum_{\pi\in\Pi_{\text{cusp}}(M)^\sigma/\text{Im}X_M} \sum_{\varphi\in\text{OB}(V(\pi)^{\mathcal{U}})} \int_{\text{Im}X_M} \langle \Phi, E(\varphi, \xi)\rangle_{\text{Pet}} E(\varphi, \xi)(g) d\xi.\end{aligned} \tag{0.1}$$

Then the righ hand side of (0.1) converges to the value $\Phi(g)$ absolutely and locally uniformly for $g \in G(\mathbb{A})$.

The proof of the convergence lemma

- For the convergence of the cuspidal part:
  - For any $R > 0$, there exists $N_1 > 0$ such that for any $\pi \in \Pi_{\text{cusp}}(G)$, any $\varphi \in \text{OB}(\pi^{\mathcal{U}})$
    $$|\varphi(g)| \ll \lambda(\pi)^{N_1} a(g)^{-R}, \quad g \in S_G.$$
  - For any $N_2 > 0$, any $\pi \in \Pi_{\text{cusp}}(G)$ and any $\varphi \in \text{OB}(\pi^{\mathcal{U}})$
    $$|\langle \Phi, \varphi \rangle_{\text{Pet}}| \leq \frac{||\Phi * \Omega^{N_2}||_2}{\lambda(\pi)^{N_2}}.$$
  - There exists $N_3 > 0$ such that
    $$\sum_{\pi \in \Pi_{\text{cusp}}(G)} \sum_{\varphi \in \text{OB}(\pi^{\mathcal{U}})} \frac{1}{\lambda(\pi)^{N_3}} < \infty.$$
    This follows from the Weyl law (by Lindenstrauss-Venkatesh): let
    $$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$$
    be the Laplacian eigenvalues on $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathcal{U})$, then
    $$\{i : \lambda_i \leq x\} \sim Cx^{\rho_0}, \quad x \to \infty.$$

$$\rightsquigarrow \sum_{\pi} \sum_{\varphi} \left| \langle \Phi, \varphi \rangle_{\text{Pet}} \varphi(g) \right| \leq ||\Phi * \Omega^{N_3}|| \cdot a(g)^{-R} \sum_{\pi} \sum_{\varphi} \frac{1}{\lambda(\pi)^{N_2 - N_1}} < \infty \quad g \in S_G.$$

- For the Eisenstein part, we apply the classification of distinguished Eisenstein series to obtain the finiteness of $\pi \in \Pi_{\text{cusp}}(M)^{\sigma}$ with $V(\pi)^{\mathcal{U}} \neq 0$.

The proof of the main result

- By the meromorphic continuation of G_s, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$,
$$H_\infty(f) = \int_{[H_2]} K_{\phi_s^{(2)} \otimes \phi_f}(t)dt$$
up to coherent distributions. Here, the Poincare series
$$K_{\phi_s^{(2)} \otimes \phi_f}(x) = \sum_{\gamma \in H_1(\mathbb{Q}) \backslash G(\mathbb{Q})} (\phi_s^{(2)} \otimes \phi_f)(\gamma x), \quad \phi_f \in \mathcal{S}(H_1(\mathbb{A}_f) \backslash G(\mathbb{A}_f))$$
- We decompose $\phi_s^{(2)}$ into two functions
$$\phi_s^{(2)} = \phi_{s,0}^{(2)} + \phi_{s,\infty}^{(2)}$$
where
    - $\phi_{s,0}^{(2)} \in C^\infty(H_1(\mathbb{R}) \backslash G(\mathbb{R}) - H_1(\mathbb{R}) U_\infty / U_\infty)$ with compact support in $H_1(\mathbb{R}) \backslash G(\mathbb{R})$
    - $\phi_{s,\infty}^{(2)} \in C^\infty(H_1(\mathbb{R}) \backslash G(\mathbb{R}) / U_\infty)$ with the same "large time behaviour" as $\phi_s^{(2)}$.
- Consider the associated Poincare seires $K_{\phi_{s,?}^{(2)} \otimes \phi_f}$ with $? = 0, \infty$.
- As $[H_2]$ is compact, the integral over $[H_2]$ for $K_{\phi_{s,0}^{(2)} \otimes \phi_f}$ can be expanded to required orbital integrals.
- Reduce to show that for $\mathrm{Re}(s)$ large enough, the Poincare series
$$K_{\phi_{s,\infty}^{(2)} \otimes \phi_f}$$
admits a (smooth) spectral expansion.

The proof of the main result

Spectral expansion of $K_{\phi^{(2)}_{s_0,\infty}\otimes\phi_f}$

Assume $n \neq 3$ and s_0 is large enough. Consider the Poincare series K_ϕ with $\phi = \phi^{(2)}_{s_0,\infty} \otimes \phi_f$. Then

$$K_\phi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathcal{U})$$

with the spectral expansion

$$K_\phi(g) = \sum_{\pi\in\Pi_{\text{cusp}}(G)^\sigma} \sum_{\varphi\in\text{OB}(\pi^{\mathcal{U}})} P^\phi_{H_1}(\overline{\varphi})\varphi(g) + \frac{\text{Vol}([H_1])}{\text{Vol}([G])}\int_{H_1(\mathbb{A})\backslash G(\mathbb{A})} \phi(g)dg$$
$$+ \sum_{\pi\in\Pi_{\text{cusp}}(M)^\sigma/\text{Im}X_M} \sum_{\varphi\in\text{OB}(V(\pi)^{\mathcal{U}})} \int_{\text{Im}X_M} P^\phi_{H_1}\left(\overline{E(\varphi,\xi)}\right) E(\varphi,\xi)(g)d\xi.$$

The right hand side is absolutely convergent and locally uniformly for $g \in G(\mathbb{A})$.

- Proof: for automorphic forms φ with certain growth condition, the Fourier coefficient
$$\langle\varphi, K_\phi\rangle = P^\phi_{H_1}(\varphi) = \int_{H_1(\mathbb{A})\backslash G(\mathbb{A})} \phi(g)P_{H_1}(R(g)\varphi)dg.$$
Now, we apply the above convergence lemma.
- If $n = 3$, $K_\phi \in L^{1+\varepsilon}(G(\mathbb{Q})\backslash G(\mathbb{A})/\mathcal{U})$, $0 \leq \varepsilon < 1$. The RSC condition implies that K_ϕ is cuspidal.