The geometry of Bernstein eigenvariety

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(Joint with C. Breuil.)

§ Eigencurves

Fix p prime, $N \geq 3$ & $p \nmid N$.

Fix $E/\mathbb{Q}_p$ fin ext'n, $\bar{\rho}: \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(k_E)$ modular Gal rep'n.

$\rightsquigarrow$ Coleman–Mazur: $\mathcal{C}_N$ rigid space $/E$

with $\mathcal{C}_{N,\bar{\rho}} \hookrightarrow (\mathrm{Spf}\, \mathbb{T}^p_{\bar{\rho}})^{rig} \times \mathbb{G}_m^{rig} \times \mathcal{W}$ $\leftarrow$ p-adic open unit ball para. characters of $\mathbb{Z}_p^\times$

$\mathbb{T}^p_{\bar\rho}$: global Hecke alg away from p.

$(\lambda, \quad a_p, \quad k)$

<u>Fact</u> For $k \in \mathbb{Z}_{\geq 0}$, $k \in \mathcal{W}$ (i.e. $(z \mapsto z^k) \in \mathcal{W}$),

$(\lambda, a_p, k) \in \mathcal{C}_N \iff \exists$ an overconvergent eigenform of level $\Gamma_0(N)$ of wt $k+2$, of $\mathbb{T}^p$-eigenvalue λ, of $\mathcal{U}^p$-eigenvalue $a_p \neq 0$.

<u>Fact</u> $\exists$ natural map

$$(\mathrm{Spf}\, \mathbb{T}^p_{\bar{\rho}})^{rig} \times \mathbb{G}_m^{rig} \times \mathcal{W} \longrightarrow (\mathrm{Spf}\, R_{\bar{\rho}})^{rig} \times \hat{T}$$

$$T = \begin{pmatrix} \mathbb{Q}_p^\times & \\ & \mathbb{Q}_p^\times \end{pmatrix}, \quad \delta_p = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \quad \hat{T} \simeq (\mathbb{G}_m^{rig})^{\oplus 2} \times \mathcal{W}$$

$\rightsquigarrow$

$$\begin{tikzcd}
(1\text{-dim})\ \mathcal{C}_{N,\bar{\rho}} \arrow[r, hook] \arrow[d] & (\mathrm{Spf}\, \mathbb{T}^p_{\bar{\rho}})^{rig} \times \mathbb{G}_m^{rig} \times \mathcal{W} \arrow[d] & \\
(2\text{-dim})\ \mathcal{S}_{N,\bar{\rho}} \arrow[r] & (\mathrm{Spf}\, R_{\bar{\rho}})^{rig} \times \hat{T} \arrow[d] & X_{tri}(\bar{\rho}_p) \arrow[dl] \\
& (\rho_{x,p}, \delta_x = \delta_{x,1} \otimes \delta_{x,2}) \in (\mathrm{Spf}\, R_{\bar{\rho}_p})^{rig} \times \hat{T} &
\end{tikzcd}$$

$\bar{\rho}_p = \bar{\rho}|_{G_{\mathbb{Q}_p}}$

Thm (Kisin) Let $x \in \mathcal{C}_N$ ($\leftrightarrow f$ of wt $k+2$)

$\rightsquigarrow \rho_x : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(E)$.

If $\rho_{x,p}$ is de Rham, then x is classical
(i.e. f is a classical form.)
($\rightsquigarrow \rho_x$ is geometric.)

Rmk This thm is a special case of Fontaine-Mazur Conj.

Fact (Kisin, KPX, Liu)

For "almost" all x,

$$(*) \qquad 0 \longrightarrow \mathcal{R}_E(\delta_{x,1}) \longrightarrow D_{rig}(\rho_{x,p}) \longrightarrow \mathcal{R}_E(\delta_{x,2}) \longrightarrow 0$$

the triangulation, which is a (φ,Γ)-mod over the Robba ring.

If $(*)$ holds, x is called a non-critical pt.

Easy part If x de Rham & non-critical, then x is classical.

Q How about x critical?

§ Another proof of Kisin's thm

By Breuil-Hellman-Schraen:

triangulline var

$$\begin{array}{ccc} \text{(2-dim) } S_{N,\bar\rho} & \xrightarrow{\ \iota\ } & X_{tri}(\bar\rho_p) \text{ (4-dim)} \\ \downarrow & & \cup \\ x & & Z_{dR} \text{ (2-dim'l de Rham cycle)} \end{array}$$

classical pts

satisfying $\rho_{x,p}$ de Rham $\Leftrightarrow \iota(x) \in Z_{dR}$.

Also, for patching module,

have $j : S_{N,\bar{p}} \longrightarrow S^{\mathrm{Pat}}_{N,\bar{p}} \supseteq Z_{cl}$ (2-dim'l classical cycle).

& $\quad x$ classical $\iff j(x)$ classical.

Denote $\quad X^{\mathrm{aut}}_{\mathrm{tri}}(\bar{\rho}_p) := \imath(S^{\mathrm{Pat}}_{N,\bar{p}}) \underset{\uparrow}{\subseteq} X_{\mathrm{tri}}(\bar{\rho}_p)$

as union of irred components

- $\imath(Z_{cl}) \subseteq Z_{dR}$,
- For $\imath(x) \in Z_{dR}$, $y_n \to x$ for $y_n \in Z_{dR}$, y_n non-critical.
- If $y_n \in \imath(S^{\mathrm{Pat}}_{N,\bar{p}})$, then $y_n \in Z_{cl}$.

<u>Thm</u> (BHS)

$X_{\mathrm{tri}}(\bar{\rho}_p)$ is locally irreducible at $\imath(x)$.

- This somehow implies: $y_n \in \imath(S^{\mathrm{Pat}}_{N,\bar{p}})$ (as $x \in S^{\mathrm{Pat}}_{N,\bar{p}}$)

$\Rightarrow y_n \in Z_{cl} \rightsquigarrow x \in Z_{cl}$.

Consider $G = GL_2(\mathbb{Q}_p) \supset B$ with $\mathrm{Lie}\, G = \mathfrak{g} \supseteq \mathfrak{b} = \mathrm{Lie}\, B$.

$$\rightsquigarrow \qquad (g, \psi) \longmapsto (gB, \mathrm{Ad}_g(\psi)).$$

$$\begin{tikzcd} \tilde{\mathfrak{g}} := G \times_B \mathfrak{b} \arrow[r, hook] \arrow[dr] & G/B \times \mathfrak{g} \arrow[d] \\ & \mathfrak{g} \end{tikzcd}$$

$X := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \hookrightarrow G/B \times G/B \times \mathfrak{g}$. $\rightsquigarrow$ <u>Fact</u> $X = \coprod_{w \in W_G} X_w$.

<u>§ Grothendieck-Springer resolution</u>

$$\begin{tikzcd} \widehat{X_{\mathrm{tri}}(\bar{\rho}_p)_x} \arrow[rr, dashed, "\text{``}\quad\text{''}"] & & \widehat{(X_{w_0})}_{x_{pdR}} \\ & Y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

- $y = (\rho_{y,p}, \delta_y) \in X_{tri}(\bar\rho_p)$,
- $D_{pdR}(\rho_{y,p})$ E-v.s. of 2-dim

$\nu_{pdR} \circlearrowleft$ $\mathrm{Fil}_H, \mathrm{Fil}_w \circlearrowleft$ (Fontaine's almost de Rham)

$\rightsquigarrow (\mathrm{Fil}_H, \mathrm{Fil}_w, \nu_{pdR}) =: y_{pdR}$.

$\mathrm{Fil}_H \in G/B$, $\mathrm{Fil}_w \in G/B$, $\nu_{pdR} \in \mathfrak{g}$

Thm (BHS) X_w is normal.

§ Bernstein eigenvariety

F^+, F/F^+ CM $\rightsquigarrow$ G definite unitary gp $/F^+$.

p inert in F^+ & unram in F.

$L := F^+_p$, $U^p \subseteq G(\mathbb{A}^{\infty,p}_{F^+})$.

$\rightsquigarrow$ $\hat{S}(U^p, E)_{\bar\rho} = [f : G(F^+)\backslash G(\mathbb{A}^{\infty}_{F^+})/U^p \to E \text{ cont}]$

$\hat{S}(U^p, E)_{\bar\rho} \circlearrowleft G(F^+_p) \simeq GL_n(L)$, $\circlearrowleft \Pi^\circ \leftarrow R_{\bar\rho}$

Let $P \subseteq GL_n$ parabolic, $L_P \subseteq P$ Levi.

- $J_P(\hat{S}(U^p, E)^{La}_{\bar\rho}) \circlearrowleft L_P(L)$
- Ω Bernstein components of $L_P(L)$,

$\rightsquigarrow Z_\Omega \rightsquigarrow (\mathrm{Spec}\, Z_\Omega)^{rig} \ (\longleftrightarrow \mathbb{G}_m^{rig})$

$x \in (\mathrm{Spec}\, Z_\Omega)^{rig}$, $x \longleftrightarrow \pi_x \circlearrowleft L_P(L)$

- μ dom wt of $L_P(L)$.
- $r := \dim Z_{L_P}$, $Z_0 := Z_{L_P}(\mathcal{O}_L)$.

$\rightsquigarrow |\mathcal{E}_{\Omega,\mu}(U^p, \bar\rho)| \hookrightarrow (\mathrm{Spf}\, R_{\bar\rho})^{rig} \times (\mathrm{Spec}\, Z_\Omega)^{rig} \times \hat{Z}_0$

$\mathcal{E}_{\Omega,\mu}(U^p, \bar\rho)$: $[L:\mathbb{Q}_p]\cdot r$-dim Bernstein eigenvar

(ρ, π, x)

<u>Fact</u> $(\rho, \pi, \chi) \in \mathcal{E}_{\Omega,\mu}(U^p, \bar\rho) \iff \underbrace{\pi \otimes \tilde\chi \circ \det}_{\text{rep of } L_p(L)} \hookrightarrow J_p(\hat{S}(U^p, E)^{la}_{\bar\rho}[p])$

$$\rightsquigarrow \quad \begin{tikzcd}
\mathcal{E}_{\Omega,\mu}(U^p, \bar\rho) \arrow[r, hook] & (\mathrm{Spf}\, R_{\bar\rho})^{rig} \times (\mathrm{Spec}\, Z_\Omega)^{rig} \times \hat{Z}_0 \arrow[d] \\
 & (\mathrm{Spf}\, R_{\bar\rho,p})^{rig} \times (\mathrm{Spf}\, Z_\Omega)^{rig} \times \hat{Z}_0 \\
\mathcal{E}^{pat}_{\Omega,\mu}(U^p, \bar\rho) \arrow[uu] \arrow[r] & X_{\Omega,\mathbb{Q}}(\bar\rho_p) \arrow[u]
\end{tikzcd}$$

<u>Thm</u> (Breuil-Ding)

(1) $\widehat{X_{\Omega,\mu}(\bar\rho_p)}_x$ is irred at de Rham pts.

(2) If ρ_p is de Rham, then

$J_p(\hat{S}(U^p, E)^{la}[p])$ has locally algebraic vectors for $L_p^D(L)$.