

TOPOLOGICAL RECONSTRUCTION THEOREMS FOR VARIETIES

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ABSTRACT. We study Torelli-type theorems in the Zariski topology for varieties of dimension at least 2, over arbitrary fields. In place of the Hodge structure, we use the linear equivalence relation on Weil divisors. Using this setup, we prove a universal Torelli theorem in the sense of Bogomolov and Tschinkel. The proofs rely heavily on new variants of the classical Fundamental Theorem of Projective Geometry of Veblen and Young.

For proper normal varieties over uncountable algebraically closed fields of characteristic 0, we show that the Zariski topological space can be used to recover the linear equivalence relation on divisors. As a consequence, we show that the underlying scheme of any such variety is uniquely determined by its Zariski topological space. We use this to prove a topological version of Gabriel's theorem, stating that a proper normal variety over an uncountable algebraically closed field of characteristic 0 is determined by its category of constructible abelian étale sheaves.

We also discuss a conjecture in arbitrary characteristic, relating the Zariski topological space to the perfection of a proper normal variety.

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1. INTRODUCTION

The underlying topological space $|X|$ of a smooth projective variety X over a field k is typically viewed as a rather weak invariant. For example, for a smooth projective curve X/k the topological space $|X|$ is determined by the cardinality of the set of points. As we discuss further below in Lemma 4.4.1, there are also many examples of homeomorphic smooth projective surfaces (over algebraic closures of finite fields)¹ that are not isomorphic.

The present paper is a reflection on what additional structures on $|X|$ enable one to recover the scheme X . There is a substantial literature on related questions. In particular, we mention the work of Bogomolov–Korotiaev–Tschinkel [3] and subsequent work of Zilber [25]. There is also related work of Cadoret–Pirutka [7] and Topaz [20, 21] for reconstruction from K -theory and other Galois-cohomological invariants. Finally, we mention the work of Voedvodsky [22], proving a conjecture of Grothendieck that the étale topos of a normal scheme of finite type over a finitely generated field uniquely determines the scheme.

We summarize the main results of the present paper somewhat informally, and with slightly stronger assumptions than in the body of the paper, in the following theorem:

Main Theorem (Universal Torelli, proper case). *Let X be a proper normal geometrically integral variety of dimension at least 2 over a field k .*

- A.** *If k is infinite or X is Cohen-Macaulay of dimension ≥ 3 , then X is uniquely determined as a scheme by the pair*

$$(|X|, c : X^{(1)} \rightarrow \text{Cl}(X)),$$

where $|X|$ is the underlying Zariski topological space, $\text{Cl}(X)$ is the group of Weil divisor classes, $X^{(1)}$ is the set of codimension 1 points of $|X|$, and c is the map sending a codimension 1 point of X to its divisor class.

Equivalently, X is determined as a scheme by its underlying topological space $|X|$ and the rational equivalence relation on the set of effective divisors.

- B.** *If k is an uncountable algebraically closed field of characteristic 0, then linear equivalence of divisors on X is determined by $|X|$. As a consequence of this and statement **A**, X is determined by $|X|$ alone.*

We also prove a categorical corollary.

Corollary (Topological Gabriel–Rosenberg, Theorem 5.4.1). *If X is a normal scheme of dimension at least 2 such that $\Gamma(X, \mathcal{O}_X)$ is an uncountable algebraically closed field of characteristic 0 and $X \rightarrow \text{Spec } \Gamma(X, \mathcal{O}_X)$ is proper, then X is uniquely determined by the category of constructible abelian étale sheaves on X .*

¹in fact, examples of homeomorphic surfaces over fields of different characteristics

As we briefly discuss in Section 5.4, this leads to a number of interesting questions about topological analogues of classical results: the work of Balmer [1] and the theory of Fourier–Mukai transforms primary among them.

The key to proving the Main Theorem is a rational form of the Fundamental Theorem of Projective Geometry, which we develop in Section 2.1. In Section 3.3, we leverage the incidence-definition of a Zariski open set of pencils in certain linear systems to recover the linear structure on set-theoretic rational equivalence classes using the rational fundamental theorem. This is inspired by work of Bogomolov and Tschinkel who used similar ideas to reconstruct function fields [4].

Remarks. We conclude the introduction with a few remarks.

- (1) Note that statement **A** is actually *false* in general for schemes of dimension 1 (see Proposition 4.5.1 below). Zilber’s proof using model theory [25] works only under the assumption that the constant fields of the curves are algebraically closed. It would be interesting to have a deeper understanding of the dimension 1 case, including finding a proof that uses only the types of methods we employ here.
- (2) The full statements of the main results are Theorem 3.1.12, Theorem 4.3.1.1, and Theorem 5.1.2 below. Note that the isomorphism type that is recovered is the isomorphism class of X over \mathbf{Z} , not over k . As an example, observe that the theorem implies that for $n \geq 4$, any Zariski homeomorphism of hypersurfaces in \mathbf{P}_K^n , with K a number field, that preserves degrees of divisors induces an isomorphism of the underlying \mathbf{Q} -schemes. For a complex hypersurface, statement **B** says that the degrees of divisors are uniquely determined by the Zariski topology, so that the underlying \mathbf{Q} -scheme is uniquely determined by the Zariski topological space. This is the best one could hope for: the group $\text{Gal}(K/\mathbf{Q})$ acts on \mathbf{P}_K^n by degree-preserving Zariski homeomorphisms.
- (3) The Main Theorem is a Torelli-type result in the following sense. One can think of Torelli’s theorem as a statement about adding a small amount of geometric content to the cohomology of a variety in order to distinguish distinct algebraic structures on a fixed differentiable manifold. The Main Theorem starts with the Zariski topology, which already encodes some of the algebraic structure – for example, the algebraic cycles – and adds the class group, which encodes the finest possible “cohomological relation” among divisors. Thus, we can think of the Main Theorem as applying the Torelli philosophy in reverse, whereupon it becomes universally true.

1.1. Conventions. In this paper, we freely use the theory of *projective structures*, as described in [15, 19] and summarized in [4, Section 3]. We will not recapitulate the theory here.

Given a commutative monoid $(M, +)$, we will call an equivalence relation Λ on M a *congruence relation* if for all $a, b, c, d \in M$, we have that $(a, c) \in \Lambda$ and $(b, d) \in \Lambda$ imply that $(a + c, b + d) \in \Lambda$.

For a vector space V over a field k we write $\mathbf{P}V$ for the projective space of lines in V . This convention makes the discussion of classical projective geometry easier, though it conflicts with the conventions of EGA.

Given a variety X and a divisor D , we will write $|D|$ for the classical linear system of D , that is $|D| = \mathbf{P}H^0(X, \mathcal{O}(D))$. We will write $|D|^\vee$ for the dual projective space $\mathbf{P}H^0(X, \mathcal{O}(D))^\vee$ (i.e., the space of hyperplanes in $|D|$), which is the natural target for the induced rational map $\nu_D : X \dashrightarrow |D|^\vee$. When the base field is algebraically closed, the closed points of the image of ν_D correspond to the hyperplanes $H_x \subset |D|$, where $H_x = \{E \in |D| : x \in E\}$.

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2. GENERIC FUNDAMENTAL THEOREMS OF PROJECTIVE GEOMETRY

In this section, we prove two strengthenings of the classical Fundamental Theorem of Projective Geometry, which states that linearity of a map of projective spaces can be detected simply by the preservation of lines. Our strengthenings have to do with assuming only that general lines (either a Zariski open – over infinite base fields – or a suitably high fraction of lines – over finite base fields) are known to be mapped to lines.

2.1. The fundamental theorem of definable projective geometry. Here we discuss a variant of the Fundamental Theorem of Projective Geometry in which one only knows distinguished subsets of “definable” lines in the projective structures and one still wishes to produce a semilinear isomorphism between the underlying vector spaces that induces the isomorphism on a dense open subset. In Section 3.3 and Section 4 we explain how to use this theory to reconstruct varieties.

Definition 2.1.1. A *definable projective space* is a triple (k, V, U) consisting of an infinite field k , a k -vector space V , and a subset $U \subset \mathrm{Gr}(1, \mathbf{P}(V))(k)$ which contains the k -points of a dense open subset of the space $\mathrm{Gr}(1, \mathbf{P}(V))$ of lines in the projective space $\mathbf{P}(V)$. The *dimension* of (k, V, U) is defined to be

$$\dim(k, V, U) := \dim_k V - 1.$$

In other words, a definable projective space is a projective space together with a collection of lines that are declared “definable” subject to some conditions.

Definition 2.1.2. Let k be a field and V a k -vector space. The *sweep* of a subset $U \subset \mathrm{Gr}(1, \mathbf{P}(V))(k)$, denoted $S_U(\mathbf{P}(V))$ is the set of k -points $p \in \mathbf{P}(V)$ that lie on some line parametrized by U .

2.1.3. Let (k, V, U) be a definable projective space. Then there exists a maximal subset $U^\circ \subset U$ which is the k -points of a Zariski open subset of $\text{Gr}(1, \mathbf{P}(V))$. Furthermore, (k, V, U°) is again a definable projective space. This is immediate from the definition.

Example 2.1.4. Fix a projective k -variety $(X, \mathcal{O}_X(1))$ of dimension d at least 2. Given a closed subset $Z \subset X$, we can associate the subspace $V(Z) \subset |\mathcal{O}(1)|$ of divisors that contain Z . The lines of the form $V(Z)$ give a subset of $\text{Gr}(1, |\mathcal{O}(1)|)$ (see Section 3.3). These are the definable lines we will consider.

The main goal of this section is to prove the following result.

Theorem 2.1.5. *Suppose (k_1, V_1, U_1) and (k_2, V_2, U_2) are finite-dimensional definable projective spaces of dimension at least 2. Given an injection $\varphi : \mathbf{P}(V_1) \rightarrow \mathbf{P}(V_2)$ that induces an inclusion $\lambda : U_1 \rightarrow U_2$, there is an isomorphism $\sigma : k_1 \rightarrow k_2$ and a σ -linear injective map of vector spaces $\psi : V_1 \rightarrow V_2$ such that $\mathbf{P}(\psi)$ agrees with φ on a Zariski-dense open subset of $\mathbf{P}(V_1)$ containing the sweep of (k_1, V_1, U_1°) .*

Remark 2.1.6. In Theorem 2.1.5 we can without loss of generality assume that $U_2 = \text{Gr}(1, \mathbf{P}(V_2))(k)$. However, we prefer to formulate it as above to make it a statement about definable projective spaces.

Remark 2.1.7. If either the dimensions of V_1 and V_2 are equal or we assume that $\lambda(U_1^\circ) \subset \text{Gr}(1, \mathbf{P}(V_2))$ is dense, then the map ψ is an isomorphism. In the case when the dimensions are equal this is immediate, and in the second case observe that if $V_1 \otimes_{k_1, \sigma} k_2 \subsetneq V_2$ is a proper subspace then there exists a dense open subset $W \subset \text{Gr}(1, \mathbf{P}(V_2))$ of lines which are not in the image of $\mathbf{P}(\psi)$, contradicting our assumption that $\mathbf{P}(\psi)(U_1^\circ) = \lambda(U_1^\circ)$ is dense.

Remark 2.1.8. Observe that two lines in a projective space are coplanar if and only if they intersect in a unique point. This enables us to describe the map $\mathbf{P}(\psi)$ as follows. Let $U' \subset U_1$ be any dense subset which is the k -points of a Zariski open subset of $\text{Gr}(1, \mathbf{P}(V_1))$, and let $P \in \mathbf{P}(V_1)$ be a point. Choose any line $\ell \subset \mathbf{P}(V_1)$ corresponding to a point of U' and not containing P (this is possible since U' is the points of an open subset of $\text{Gr}(1, \mathbf{P}(V_1))$), and let $Q, R \in \ell$ be two distinct points. Let $L_{P,Q}$ (resp. $L_{P,R}$) be the line through P and Q (resp. P and R), and choose points $S \in L_{P,Q} - \{P, Q\}$ and $T \in L_{P,R} - \{P, R\}$ such that the line $L_{S,T}$ through S and T is also given by a point of U' (it is possible to choose such S and T since U' is the k -points of an open set). The lines $L_{S,T}$ and $L_{Q,R} = \ell$ are then coplanar and therefore intersect in a unique point E . It follows that $\varphi(L_{S,T})$ and $\varphi(L_{Q,R})$, which are lines since $L_{S,T}$ and $L_{Q,R}$ are definable, are coplanar since they intersect in $\varphi(E)$. It follows that the lines in $\mathbf{P}(V_2)$ given by $L_{\varphi(Q), \varphi(T)}$ and $L_{\varphi(S), \varphi(R)}$ are coplanar and consequently intersect in a unique point, which is $\mathbf{P}(\psi)$.

This description will play an important role in Section 2.3 below.

Proof of Theorem 2.1.5. This proof is very similar to the proof due to Emil Artin in the classical case, as described by Jacobson in [12, Section 8.4].

We may without loss of generality assume that $U_1 = U_1^\circ$.

Let us begin by showing the existence of the isomorphism of fields $\sigma : k_1 \rightarrow k_2$. The construction will be in several steps.

First we set up some basic notation. Let V be a vector space over a field k . For a nonzero element $v \in V$ let $[v] \in \mathbf{P}(V)$ denote the point given by the line spanned by v . For $P \in \mathbf{P}(V)$ write $\ell_P \subset V$ for the line corresponding to P , and for two distinct points $P, Q \in \mathbf{P}(V)$ write $L_{P,Q} \subset \mathbf{P}(V)$ for the projective line connecting P and Q . If $P = [v]$ and $Q = [w]$ then $L_{P,Q}$ corresponds to the 2-dimensional subspace of V given by

$$\text{Span}(v, w) := \{av + bw \mid a, b \in k\}.$$

If $L \subset \mathbf{P}(V)$ is a line and $P, Q, R \in L$ are three pairwise distinct points then there is a unique k -linear isomorphism $L \xrightarrow{\sim} \mathbf{P}^1$ sending P to 0, Q to 1, and R to ∞ . For a collection of data $(L, \{P, Q, R\})$ we therefore have a canonical identification

$$\epsilon^{P,Q,R} : k \xrightarrow{\sim} L - \{R\}.$$

In the case when $L = L_{[v],[w]}$ for two non-colinear vectors $v, w \in V - \{0\}$ we take $P = [v]$, $Q = [v + w]$, and $R = [w]$. Then the identification of k with $L - \{R\}$ is given by

$$a \mapsto v + aw.$$

Suppose given $(L, \{P, Q, R\})$ as above, and fix a basis vector $v_P \in \ell_P$. Then one sees that there exists a unique basis vector $v_R \in \ell_R$ such that $[v_P + v_R] = Q$. This observation enables us to relate the maps $\epsilon^{P,Q,R}$ for different lines as follows.

Consider a second line L' passing through P and equipped with two additional points $\{S, T\}$, and let $a, b \in k - \{0\}$ be two scalars. We can then consider the two lines

$$L_{T,R}, \quad L_{\epsilon^{P,Q,R}(a), \epsilon^{P,S,T}(b)},$$

which will intersect in some point

$$\{O\} = L_{T,R} \cap L_{\epsilon^{P,Q,R}(a), \epsilon^{P,S,T}(b)}.$$

The situation is summarized in the following picture, where to ease notation we write simply a (resp. b) for $\epsilon^{P,Q,R}(a)$ (resp. $\epsilon^{P,S,T}(b)$):

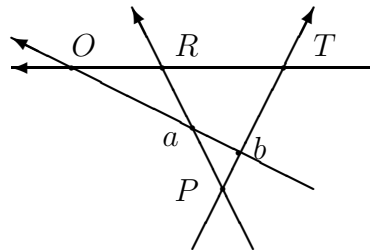


Figure 1

If we fix a basis element $v_P \in \ell_P$ we get by the above observation a basis vector v_Q (resp. v_R, v_S, v_T) for ℓ_Q (resp. ℓ_R, ℓ_S, ℓ_T), which in turn gives an identification

$$\epsilon^{[v_T],[v_T+v_R],[v_R]} : k \xrightarrow{\sim} L_{T,R} - \{R\}.$$

An elementary calculation then shows that

$$O = \epsilon^{[v_T],[v_T+v_R],[v_R]}(-a/b).$$

In particular, if $a = b$ then the point O is independent of the choice of a , and furthermore it follows from the construction that O is also independent of the choice of the basis element v_P .

Consider now a definable projective space (k, V, U) , and let $L_0 \subset \mathbf{P}(V)$ be a definable line with three points $P, Q, R \in L_0$. Fix $a \in k$ so we have a point

$$\epsilon^{P,Q,R}(a) \in L_0.$$

Let M_P denote the scheme classifying data $(L, \{S, T\})$, where L is a line through P and $\{S, T\}$ is a set of two additional points on L such that P, S , and T are all distinct. The scheme M_P has the following description. The point P corresponds to a line $\ell_P \subset V$ and the set of lines passing through P is given by $\mathbf{P}(V/\ell_P)$. If $\mathcal{L} \rightarrow \mathbf{P}(V/\ell_P)$ denotes the universal line in $\mathbf{P}(V)$ passing through P then there is an open immersion

$$M_P \subset \mathcal{L} \times_{\mathbf{P}(V/\ell_P)} \mathcal{L},$$

whence M_P is smooth, geometrically connected, and rational. Since k is infinite it follows that the k -points of M_P are dense.

Lemma 2.1.9. *Fix $a \in k$. There exist a nonempty open subset $U_{P,a} \subset M_P$ such that if $(L, \{S, T\})$ is a line through P with two points corresponding to a k -point of $U_{P,a}$ then the lines*

$$(2.1.9.1) \quad L_{P,T}, \quad L_{T,R}, \quad L_{\epsilon^{P,Q,R}(a), \epsilon^{P,S,T}(a)}$$

are all definable.

Proof. We may without loss of generality assume that $U = U^\circ$.

Let $Q_0 \in M_P$ denote the point corresponding to $(L_0, \{Q, R\})$. The procedure of assigning one of the lines in (2.1.9.1) to a pointed line $(L, \{S, T\})$ is a map

$$q : M_P \rightarrow \mathrm{Gr}(1, \mathbf{P}(V)).$$

Note that the image of this map contains the point corresponding to the line L_0 , and therefore the inverse image $q^{-1}(U)$ is nonempty. Since M_P is integral it follows that the intersection of the preimages of U under the three maps defined by (2.1.9.1) is nonempty. \square

A variant of the above lemma is the following, which we will use below.

Lemma 2.1.10. *With notation as in Lemma 2.1.9, let $P, Q \in \mathbf{P}(V)$ be two points in the sweep of U° . Then there exists a definable line L_P through P and a definable line L_Q through Q such that L_P and L_Q intersect in a point R .*

Proof. Let $N_P \subset \mathrm{Gr}(1, \mathbf{P}(V))$ denote the space of lines through P , so $N_P \simeq \mathbf{P}(V/\ell_P)$ for the line $\ell_P \subset V$ corresponding to P . Let $\mathcal{L} \rightarrow N_P$ denote the universal line through P , and let $s : N_P \rightarrow \mathcal{L}$ denote the tautological section. Then the natural map

$$\mathcal{L} - \{s(N_P)\} \rightarrow \mathbf{P}(V) - \{P\}$$

is an isomorphism, since any two distinct points lie on a unique line. The set of points of $\mathbf{P}(V) - \{P\}$ which can be connected to P by a line given by a point of U° is under this isomorphism identified with the preimage of $U^\circ \cap N_P$. In particular, this set is nonempty and open. It follows that the set of points of $\mathbf{P}(V)$ which can be connected

to both P and Q by lines given by points of U° is the intersection of two dense open subsets, and therefore is nonempty. \square

With these preparations we can now proceed with the proof of Theorem 2.1.5. Proceeding with the notation of the theorem, let us first define the map $\sigma : k_1 \rightarrow k_2$. Choose a definable line $L_0 \subset \mathbf{P}(V_1)$ together with three points $P, Q, R \in L_0$ such that $\varphi(L_0) \subset \mathbf{P}(V_2)$ is also a definable line. We then get a map

$$k_1 \xrightarrow{\epsilon^{P,Q,R}} L_0 - \{R\} \xrightarrow{\varphi} \varphi(L_0) - \{\varphi(R)\} \xrightarrow{(\epsilon^{\varphi(P),\varphi(Q),\varphi(R)})^{-1}} k_2,$$

which we temporarily denote by $\sigma^{(L_0, \{P, Q, R\})}$.

Claim 2.1.11. *The map $\sigma^{(L_0, \{P, Q, R\})}$ is independent of $(L_0, \{P, Q, R\})$.*

Proof. Let $(L'_0, \{P', Q', R'\})$ be a second definable line with three points. Given $a \in k_1$, we will show that

$$\sigma^{(L_0, \{P, Q, R\})}(a) = \sigma^{(L'_0, \{P', Q', R'\})}(a).$$

From the definition, we see that this holds for $a = 0$ and $a = 1$, so we assume that $a \neq 0$ in what follows. First consider the case when $P = P'$. By Lemma 2.1.9 we can find a line L with two points $\{S, T\}$ such that the lines (2.1.9.1) are all definable, as well as the lines (2.1.9.1) obtained by replacing $(L_0, \{P, Q, R\})$ with $(L'_0, \{P, Q', R'\})$

The picture in Figure 1 is taken by φ to the corresponding picture in $\mathbf{P}(V_2)$. Looking at the intersection point it follows that

$$\sigma^{(L_0, \{P, Q, R\})}(a) = \sigma^{(L, \{P, S, T\})}(a) = \sigma^{(L'_0, \{P, Q', R'\})}(a).$$

It follows, in particular, that the map $\sigma^{(L_0, \{P, Q, R\})}$ is independent of the points Q and R . Since $\sigma^{(L_0, \{R, Q, P\})}$ is given by the formula

$$\iota_{k_2} \circ \sigma^{(L_0, \{P, Q, R\})} \circ \iota_{k_1},$$

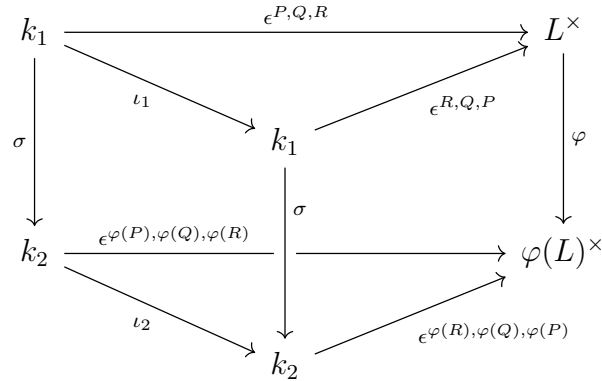
where ι_{k_j} denotes the involution of k_j^* given by $u \mapsto u^{-1}$, it follows that the map $\sigma^{(L_0, \{P, Q, R\})}$ is independent of the triple $\{P, Q, R\}$, so we get a well-defined map $\sigma^{L_0} : k_1 \rightarrow k_2$. Now for a second definable line L'_0 which has nonempty intersection with L_0 the intersection, the point $P := L_0 \cap L'_0$ is on both lines so we can apply the preceding discussion with the two lines L_0 and L'_0 and Q, R and Q', R' chosen arbitrarily to deduce the independence of the choice of $(L_0, \{P, Q, R\})$. Finally for an arbitrary definable line we can by Lemma 2.1.10 find a chain (in fact of length 2) of definable lines which connect the two, which concludes the proof. \square

Let us write the map of Claim 2.1.11 as $\sigma : k_1 \rightarrow k_2$.

Claim 2.1.12. *The map σ is an isomorphism of fields.*

Proof. First note that by construction the map σ sends 1 to 1 and is compatible with the inversion map $a \mapsto a^{-1}$. Indeed the statement that $\sigma(1) = 1$ is immediate from the construction and the compatibility with the inversion map can be seen as follows. Let $\iota_j : k_j^\times \rightarrow k_j^\times$ ($k = 1, 2$) denote the map $a \mapsto a^{-1}$, and let $(L, \{P, Q, R\})$ be a definable line with three marked points. Write L^\times (resp. $\varphi(L)^\times$) for $L - \{P, R\}$ (resp. $\varphi(L) -$

$\{\varphi(P), \varphi(R)\}$). Then by the independence of the choice of marked line in the definition of σ , we have that the diagram



commutes. The compatibility with the multiplicative structure again follows from contemplating Figure 1, and the observation that by construction the map σ takes 1 to 1. Indeed given $a, b \in k_1^\times$ such that all the lines in Figure 1 are definable, we must have

$$(2.1.12.1) \quad \sigma(-a/b) = -\sigma(a)/\sigma(b)$$

since this fraction is given by the point O . Since the condition of being definable is open (by our initial reduction to $U_1 = U_1^\circ$), the fact that for any definable $(L, \{P, Q, R\})$ the line through $\epsilon^{P,Q,R}(a)$ and $\epsilon^{P,Q,R}(b)$ is definable implies that the same is true after deforming $(L, \{P, Q, R\})$. Thus we get the formula (2.1.12.1) for all a and b . In particular, taking $b = 1$ we get that $\sigma(-a) = -\sigma(a)$ for all a , and since σ is compatible with the inversion maps we get that

$$\sigma(ab) = \sigma(a)\sigma(b)$$

for all $a, b \in k^\times$. Since 0 is also taken to 0 by σ we in fact get this formula for all $a, b \in k$.

For the verification of the compatibility with additive structure, consider a marked line $(L, \{P, Q, R\})$. Let S be a point not on the line and let T be a third point on $L_{S,R}$. The lines $L_{P,T}$ and $L_{Q,S}$ intersect in a point we call V , and then the line $L_{V,R}$ intersects $L_{P,S}$ in a point we call W . This is summarized in the following picture, where we write simply a (resp. b) for $\epsilon^{P,Q,R}(a)$ (resp. $\epsilon^{S,T,R}(b)$).

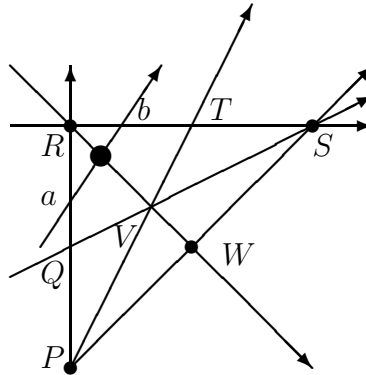


Figure 2

A straightforward calculation done by choosing a basis $v_R \in \ell_R$ then shows that the point of intersection marked with the larger bullet is the point

$$\epsilon^{W,V,R}(a+b).$$

To prove that σ is compatible with the additive structure it suffices to show the following lemma, which concludes the proof. \square

Lemma 2.1.13. *For any $a, b \in k$ there exists a pointed line $(L, \{P, Q, R\})$ and points S and T such that all the lines in Figure 2 are definable.*

Proof. The collections of data

$$(2.1.13.1) \quad (L, \{P, Q, R\}, \{S, T\})$$

defining a diagram as in Figure 2 are classified by an irreducible scheme M , each line in the diagram gives a morphism

$$t : M \rightarrow \text{Gr}(1, \mathbf{P}(V_1)).$$

It therefore suffices to show that for any particular choice of line in Figure 2, there exists a choice of (2.1.13.1) for which that line is definable. Indeed, then the set of choices of data (2.1.13.1) for which that line is definable is nonempty and open in M . Since the M is irreducible the intersection of nonempty open sets is nonempty and we conclude that there exists a point for which all the lines in Figure 2 are definable.

For the line through R, V , and W this follows from noting that the data of the colinear points S and T is equivalent to the data of the points $\{V, W\}$. Indeed given these two colinear points, the lines \overline{QV} and \overline{PQ} are coplanar and therefore intersect in a unique point, which defines S , and the intersection of \overline{SR} and \overline{PV} then defines T . Therefore the map t is smooth and dominant in this case, so the preimage of U_1 is nonempty.

For the other lines in Figure 2, note that we can extend the map t to the bigger (but still irreducible) scheme \overline{M} classifying collections of data $(L, \{P, Q, R\}, \{S, T\})$, where as before L is a point, $\{P, Q, R\}$ are three points on L , and $\{S, T\}$ are two additional points which are colinear with R , but where we no longer insist that the line through T and S is distinct from L , but only that the points $\{P, Q, R, S, T, a, b\}$ are distinct. Now it is clear that the preimage in \overline{M} of U_1 is nonempty since we can take all the points to lie on the same definable line L . \square

Now that we have constructed the isomorphism σ , it remains to construct the map $\psi : V_1 \rightarrow V_2$.

First note that we can choose a basis e_1, \dots, e_n for V_1 with the property that the span of e_i and e_j is a definable line for any $i \neq j$. Define $e'_1, \dots, e'_n \in V_2$ as follows. For e'_1 we take any basis element in $\ell_{\varphi([e_1])}$. Now for each $e_i, i \geq 2$, the line in $\mathbf{P}(V_1)$ associated to the plane $\text{Span}(e_1, e_i)$ is definable, and therefore the image under φ is a definable line and contains the points $\varphi([e_1]), \varphi([e_i]),$ and $\varphi([e_1 + e_i])$. The choice of the representative e'_1 for $\varphi([e_1])$ defines a representative e'_i for $\varphi([e_i])$ such that $\varphi([e_1 + e_i]) = e'_1 + e'_i$. Consider the map

$$\gamma : V_1 \rightarrow V_2$$

defined by

$$\gamma(a_1 e_1 + \dots + a_n e_n) := \sigma(a_1) e'_1 + \dots + \sigma(a_n) e'_n.$$

Claim 2.1.14. For general (a_1, \dots, a_n) we have

$$\varphi([a_1e_1 + \dots + a_n e_n]) = [\gamma(a_1e_1 + \dots + a_n e_n)].$$

Proof. By the construction of σ , if for each $2 \leq i \leq n$ the vectors

$$(2.1.14.1) \quad a_1e_1 + \dots + a_{i-1}e_{i-1}, \quad a_i e_i$$

span a definable line, then we get by induction on i that

$$\varphi([a_1e_1 + \dots + a_i e_i]) = [\gamma(a_1e_1 + \dots + a_i e_i)].$$

Now for each i the map sending a vector (a_1, \dots, a_n) to the span of the elements (2.1.14.1) defines a map

$$A \rightarrow \mathbf{G}(1, \mathbf{P}(V_1))$$

whose image meets U_1 . Taking the common intersection of the preimages of U_1 under these maps, we get a nonempty open subset $A^\circ \subset A$ of tuples $(a_1, \dots, a_n) \in A(k_1)$ for which the vectors (2.1.14.1) span a definable line. \square

As a consequence, the map γ defined above is uniquely associated to φ , up to scalar, and is thus independent of the general choice of basis e_1, \dots, e_n .

To complete the proof of Theorem 2.1.5 it suffices to show that $\mathbf{P}(\gamma)$ agrees with φ on the entire sweep of (k_1, V_1, U_1) . By the above remark, to show this for a particular point p , it suffices to work with any general basis. To prove this we show that given a point $p \in S_{U_1}(\mathbf{P}_{k_1}(V_1))$ there exists a basis e_1, \dots, e_n for V_1 as above for which p lies in the resulting subset A° . Reviewing the above construction, one sees that it suffices to show that we can find a basis e_1, \dots, e_n for V_1 such that the following hold:

- (i) p is the point corresponding to the line spanned by e_1 .
- (ii) Any two elements e_i and e_j , with $i \neq j$, span a definable line.
- (iii) For any $2 \leq i \leq n$ the vectors

$$e_1 + \dots + e_{i-1}, \quad e_i$$

span a definable line.

For this start by choosing e_1 so that (i) holds. Since p lies in the sweep we can then find e_2 such that e_1 and e_2 span a definable line. Now observe that given $2 \leq r \leq n$ and a basis e_1, \dots, e_r satisfying (ii) and (iii) with $i, j \leq r$ we can find e_{r+1} such that (ii) and (iii) hold with $i, j \leq r+1$. Indeed a general choice of vector in V_1 will do for e_{r+1} since for given fixed vector v_0 lying in the sweep there is a nonempty Zariski open subset of vectors w such that w and v_0 span a definable line.

This completes the proof of the Theorem. \square

2.2. A variant fundamental theorem. Suppose (k_1, V_1, U_2) and (k_2, V_2, U_2) are finite-dimensional definable projective spaces. Write $P_i = \mathbf{P}_{k_i}(V_i)$ for the associated projective space for $i = 1, 2$.

In this section we prove the following result, weakening the assumptions of Theorem 2.1.5. This is included primarily for technical reasons related to Section 3.3; a reader interested in working only over algebraically closed fields can ignore this section on a first reading.

Theorem 2.2.1. *Assume P_1 and P_2 have dimension at least 2. Suppose $\sigma : P_1 \rightarrow P_2$ is a bijection such that each line in U_1 is sent under σ to a linear subspace of P_2 and each line in U_2 is sent under σ^{-1} to a linear subspace of P_1 . Then σ sends elements of U_1 to lines and it agrees with a linear isomorphism $P_1 \rightarrow P_2$ on the sweep of U_1 .*

Without loss of generality, we assume that U_i is the k_i -points of an open subset of the appropriate Grassmannian, and we make this assumption for the remainder of the proof.

The proof of Theorem 2.2.1 appears at the very end of this section, after several requisite precursors about data (P_1, P_2, σ) as in the theorem are developed (assuming U_1 and U_2 are the k -points of open subsets).

Remark 2.2.2. Note that in the statement of Theorem 2.2.1, we only assume that lines are sent to *linear subspaces*, not to lines. This comes up naturally when one seeks to define lines in linear systems using incidence relations: given a subset Z of a variety X , the set of sections of a linear system that contains Z is a linear subspace. Detecting the dimension of this linear subspace is quite subtle when the field of constants of X is not algebraically closed. In particular, while it may be obvious that such a subset is a line on one side of an isomorphism, it is not generally clear that it remains a line on the other side. A trivial (and not particularly informative) example comes from the existence of an abstract bijection between a line and a projective space of arbitrary positive dimension. As we will see in the proof, one needs control over a large set of lines to avoid this situation.

Definition 2.2.3. A pair $(D_0, D_1) \in P_1^2$ is *good* if it lies in the inverse image of U_1 under the natural span map $P_1^2 \setminus \Delta \rightarrow \text{Gr}(1, P_1)$.

Definition 2.2.4. A collection of elements

$$\mathbf{D} := (D_0, \dots, D_s)$$

of P_1 is *admissible* if for any two $0 \leq i, j \leq s$ the pair (D_i, D_j) is good and if the D_i span a linear subspace of P_1 of dimension s .

We fix an admissible collection \mathbf{D} in what follows.

Definition 2.2.5. A point $Q \in P_1$ is *\mathbf{D} -good* if (D_i, Q) is good for all i .

For a pair i, j let $\ell_{ij}^1 \subset P_1$ be the line spanned by D_i and D_j and let $\ell_{ij}^2 \subset P_2$ denote the line spanned by $\sigma(D_i)$ and $\sigma(D_j)$. Note that since ℓ_{ij}^1 is definable, which implies that $\sigma(\ell_{ij}^1)$ is a linear subspace $T_{ij}^2 \subset P_2$ containing $\sigma(D_i)$ and $\sigma(D_j)$, we have

$$\ell_{ij}^2 \subset T_{ij}^2.$$

2.2.6. For $t \leq s$ define $W_t^2 \subset P_2$ inductively as follows. For $t = 0$ we define $W_0^2 := \sigma(D_0)$. Then inductively define W_{t+1}^2 to be the linear span of W_t^2 and $\sigma(\ell_{t,t+1})$. When we want to be unambiguous, we will write $W_t^2(\mathbf{D})$ to denote the dependence upon \mathbf{D} . Note that it is *a priori* possible for \mathbf{D} and \mathbf{D}' to have the same span in P_1 while $W_s^2(\mathbf{D}) \neq W_s^2(\mathbf{D}')$.

Let $Q \in P_1$ be a \mathbf{D} -good point.

Theorem 2.2.7. *If $\sigma(Q) \in W_s^2(\mathbf{D})$ then Q is in the linear span of the D_i .*

2.2.8. We first identify mutations of \mathbf{D} that leave Theorem 2.2.7 invariant. In each of the following two cases we have that if the assumptions hold for \mathbf{D} then they hold after replacing \mathbf{D} by \mathbf{D}' and if the conclusions hold for \mathbf{D}' then they hold for \mathbf{D} , and therefore in the proof we may replace \mathbf{D} by \mathbf{D}' .

- I. Suppose \mathbf{D}' is an admissible tuple gotten by replacing D_s by a point $D'_s \in \ell_{s-1,s}$ such that Q is \mathbf{D}' -good and $\sigma(D'_s)$ lies in the linear span $\langle W_{s-1}^2(\mathbf{D}), \sigma(Q) \rangle$. Then we have that $\sigma(Q) \in W_s^2(\mathbf{D}')$ and Q is \mathbf{D}' -good. Moreover, we have that $\langle \mathbf{D} \rangle = \langle \mathbf{D}' \rangle$.
- II. Suppose \mathbf{D}' is an admissible tuple gotten by replacing D_{s-1} by a point $D'_{s-1} \in \ell_{s-1,s}$ such that Q is \mathbf{D}' -good and $\sigma(D_{s-1}) \in \langle \sigma(D_s), \sigma(Q) \rangle$.

These mutations will arise as follows: the set of choices D'_s or D'_{s-1} will range through a line contained in the definable subspace $\sigma(\ell_{s-1,s})$. Since the base field is infinite, such a line is infinite, so its preimage in the line $\ell_{s-1,s}$ hits every open subset. This is main way in which we use the fact that the definable subspaces $\ell_{i-1,i}$ are lines.

2.2.9. We assume that $\sigma(Q) \in W_s^2$ and show that Q is in the linear span of the D_i .

The basic idea is to work inductively by projection from D_s to the lower dimensional subspace. To get things into appropriately general position, however, we will do this along with modifying our original configuration (D_0, \dots, D_s) so as to obtain a contradiction.

First of all, by our assumptions the line ℓ through Q and D_s is definable, so $\sigma(\ell) \cap W_s^2$ is a linear subspace of positive dimension.

Furthermore, proceeding by induction we may assume that the theorem holds for collections of elements (D_0, \dots, D_t) with $t < s$. Note here that the statement for $s = 0$ is trivial.

Lemma 2.2.10. *The following hold.*

- (i) $\sigma(\ell_{s-1,s}) \cap W_{s-1}^2 = \{\sigma(D_{s-1})\}$.
- (ii) *The intersection of $\sigma(\ell_{s-1,s})$ with the linear span of W_{s-1}^2 and $\sigma(Q)$ is a positive dimensional linear subspace of $\sigma(\ell_{s-1,s})$.*

Proof. Note that all but finitely many points $U \in \ell_{s-1,s}$ the collection

$$(D_0, \dots, D_{s-1}, U)$$

is admissible, and U does not lie in the linear span of (D_0, \dots, D_{s-1}) (recall that U_1 is assumed open). By the induction hypothesis it follows that the intersection

$$\sigma(\ell_{s-1,s}) \cap W_{s-1}^2$$

is finite. Since this is also a linear space it follows that it consists of exactly one point, namely $\sigma(D_{s-1})$. This proves (i).

For (ii), let δ be the dimension of the linear space $\sigma(\ell_{s-1,s})$. Then using (i) we have

$$\dim(W_{s-1}^2) + \delta = \dim(W_s^2).$$

On the other hand, the dimension of the linear span of $\sigma(Q)$ and W_{s-1}^2 is equal to

$$\dim(W_{s-1}^2) + 1.$$

Therefore the intersection in question in (ii) is given by intersection a δ -dimensional space with a $\dim(W_{s-1}^2) + 1$ -dimensional space inside a $\dim(W_{s-1}^2) + 1$ -dimensional space. From this (ii) follows. \square

2.2.11. For all but finitely many points $D'_s \in \ell_{s-1,s}$ the collection of elements

$$\mathbf{D}' := (D_0, \dots, D_{s-1}, D'_s)$$

is admissible and each of these elements is pairwise good with Q . We can therefore find $D'_s \in \ell_{s-1,s}$ such that \mathbf{D}' is admissible, Q is \mathbf{D}' -good, and $\sigma(D'_s)$ lies in the linear span of $W_{s-1}^2(\mathbf{D}')$ and $\sigma(Q)$. Replacing \mathbf{D} by such a \mathbf{D}' is an allowable mutation of type I, as in Paragraph 2.2.8.

2.2.12. Consider the projection

$$q_s : \langle \sigma(D_s), W_{s-1}^2 \rangle \dashrightarrow W_{s-1}^2$$

from the linear span of $\sigma(D_s)$ and W_{s-1}^2 sending an element R to the intersection of the line through $\sigma(D_s)$ and R with W_{s-1}^2 . This is defined in a neighborhood of the line through $\sigma(D_{s-1})$ and $\sigma(Q)$. In particular, it is defined at $\sigma(Q)$.

Let $Q_{s-1} \in W_{s-1}^2$ denote $\sigma^{-1}(q_s(\sigma(Q)))$. (We will write $Q_{s-1}(\mathbf{D})$ when we want to remember the dependence on \mathbf{D} .)

2.2.13. If $\sigma(Q_{s-1}) \in W_{s-2}^2$ then we are done by applying our induction hypothesis to

$$(D_0, \dots, D_{s-2}, D_s)$$

and Q .

So assume $\sigma(Q_{s-1})$ is not in W_{s-2}^2 .

2.2.14. The space W_{s-1}^2 is the linear span of W_{s-2}^2 and $\sigma(\ell_{s-1,s-2})$. Since the space spanned by W_{s-2}^2 and $\sigma(Q_{s-1})$ is assumed strictly bigger than W_{s-2}^2 , this space meets a line $T \subset \sigma(\ell_{s-1,s-2})$. For all but finitely many elements $D'_{s-1} \in \ell_{s-1,s-2}$ the collection

$$\mathbf{D}'' := (D_0, \dots, D_{s-2}, D'_{s-1}, D_s)$$

is again admissible. Replacing D_{s-1} by a suitable element of $\sigma^{-1}(T)$ we may therefore further assume that the line through $\sigma(D_{s-1})$ and $\sigma(Q_{s-1})$ meets W_{s-2}^2 . Call this point $R_0 \in W_{s-2}^2$. (We will write $R_0(\mathbf{D})$ when we want to remember the dependence on \mathbf{D} .)

Note that the construction ensures that

$$\sigma(Q_{s-1}(\mathbf{D})) = \langle \sigma(D_{s-1}), R_0(\mathbf{D}) \rangle \cap \langle \sigma(D_s), \sigma(Q) \rangle$$

for any \mathbf{D} satisfying the assumptions.

2.2.15. We claim that for a suitably chosen element $D'_{s-1} \in \ell_{s-1,s}$ we can arrange that Q_{s-1} is \mathbf{D}'' -good.

The linear span of $\sigma(\ell_{s-1,s})$ and R_0 contains the line connecting $\sigma(D_s)$ and $\sigma(Q_{s-1})$. It follows that the plane spanned by R_0 , $\sigma(Q)$, and $\sigma(Q_{s-1})$ meets $\sigma(\ell_{s-1,s})$ in a line M .

Observe that R_0 does not lie in M . To see this note that if that was the case then $R_0 \in \sigma(\ell_{s-1,s})$ which would imply $\sigma(Q_{s-1}) \in \sigma(\ell_{s-1,s})$, which would imply $\sigma(Q) \in \sigma(\ell_{s-1,s})$ so Q is in $\ell_{s-1,s}$, which we are assuming is not the case.

We then have an infinitude of elements $D'_{s-1} \in \sigma^{-1}(M)$ such that the collection

$$\mathbf{D}'' := (D_0, \dots, D'_{s-1}, D_s)$$

is admissible.

Keeping R_0 fixed, we see that the set of points of the form

$$\sigma^{-1}(\langle \sigma(D'_{s-1}), R_0 \rangle \cap \langle \sigma(D_s), \sigma(Q) \rangle)$$

is an infinite subset of the definable line $\langle D_s, Q \rangle$. Since D'' is admissible, any such infinite subset contains infinitely many points that are D'' -good, as desired.

Replacing D with D'' is an allowable mutation of type II as in Paragraph 2.2.8. Therefore for suitable chosen $D'_{s-1} \in \sigma^{-1}(M)$ we get that Q_{s-1} is D'' -good.

2.2.16. Applying our induction hypothesis we conclude that Q_{s-1} is in the linear span of D_0, \dots, D_{s-1} . Since $\sigma(Q)$ lies in the line $\langle \sigma(D_s), \sigma(Q_{s-1}) \rangle$, we have that $\sigma(Q) \in \sigma(\langle D_s, Q_{s-1} \rangle)$, and thus Q lies in the definable line $\langle D_s, Q_{s-1} \rangle$.

This completes the proof of Theorem 2.2.7. \square

Corollary 2.2.17. *Let (D_0, \dots, D_s) be an admissible collection of elements of P_1 . Then we have*

$$s \leq \dim W_s^2 = \sum_{i=1}^s \dim \sigma(\ell_{i-1,i}),$$

with equality if and only if each space $\sigma(\ell_{i-1,i})$ is a line.

Proof. This follows immediately from Lemma 2.2.10(i). Note that this lemma uses the induction hypothesis of the proof of Theorem 2.2.7, but with the proof of that theorem complete we can apply the lemma unconditionally. \square

Corollary 2.2.18. *The dimensions of P_1 and P_2 are equal and for a good pair of points (E, F) in P_1 the line through E and F is sent under σ to a line in P_2 .*

Proof. By the preceding corollary we see that the dimension of P_2 is at least that of the dimension of P_1 , and by consideration of σ^{-1} we see that they must be equal.

With the equality of dimensions established, note that if (E, F) is a good pair that spans a line ℓ such that the dimension of $\sigma(\ell)$ is > 1 , then we can extend (E, F) to an admissible collection

$$D = (D_0, \dots, D_{\dim(P_1)})$$

with

$$(D_0, D_1) = (E, F)$$

and Corollary 2.2.17 gives

$$\dim W_{\dim(P_1)}^2 > \dim(P_1)$$

contradicting the equality of dimensions. \square

Proof of Theorem 2.2.1. This follows from Corollary 2.2.18 and Theorem 2.1.5. \square

2.3. The probabilistic fundamental theorem of projective geometry. In this section, we prove that knowing most lines also determines linearity of a map of *finite* projective spaces.

To state the main result consider the following functions of three variables (whose origin will be explained in the proof):

$$(2.3.0.1) \quad A(q, n, \epsilon) := 2 \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right) \cdot \left(\frac{q^{n+1}-q}{q^{n+1}-1} \right)^{-2} - \frac{(q-1)^2}{q^{n+1}-1}$$

(2.3.0.2)

$$B(q, n, \epsilon) := 2(q-1) \cdot \frac{q^{n+1} - 1}{q^{n+1} - q} \cdot \left(2\epsilon + 2A(q, n, \epsilon) + 2 \frac{q^2(q+1)^2}{q^{n+1} - q} \right) + 2A(q, n, \epsilon) + 2 \frac{q^2(q+1)^2}{q^{n+1} - q}.$$

The main result of this section is the following:

Theorem 2.3.1. *Let \mathbb{F} be a finite field with q elements, and let P_1 and P_2 be projective spaces over \mathbb{F} of dimension $n > 3$. Let $f : P_1 \rightarrow P_2$ be an injection of sets. Assume given $\epsilon > 0$ such that the proportion of lines $L \subset P_1$ for which $f(L) \subset P_2$ is a line is at least $1 - \epsilon$, and assume that*

$$A(q, n, \epsilon) + 2 \frac{q^2(q+1)^2}{q^{n+1} - 1} < \frac{1}{2}$$

and

$$9B(q, n, \epsilon) + q^{-n+3} < 1.$$

Then there is an injection $f' : P_1 \rightarrow P_2$ that takes lines to lines, and such that the proportion of elements of P_1 on which f and f' agree is at least

$$1 - 2\epsilon - 2A(q, n, \epsilon) - 2 \frac{q^2(q+1)^2}{q^{n+1} - q}.$$

Remark 2.3.2. Theorem 2.3.1 is similar in spirit to the Blum-Luby-Rubinfeld linearity test [2, Lemmas 9-12], part of the theory of property testing in computer science. The arguments of that paper show that given a function $f : G \rightarrow G'$ for groups G, G' , if the proportion of $x, y \in G$ such that $f(x)f(y) = f(xy)$ is close enough to 1, then there exists a group homomorphism $f' : G \rightarrow G'$ such that $f(x) = f'(x)$ for a proportion of x close to 1. Their methods are computational and give an approach to find f' . Theorem 2.3.1 solves the analogous problem where, instead of a group of homomorphism, we have an injective map of projective spaces of large enough dimension. The strategy of [2] relies on choosing $f'(x)$ so that $f'(x) = f(xy^{-1})f(y)$ for a proportion of y close to 1, and our strategy uses a similar, but more complex, formula adapted to the case of projective spaces.

After building up suitable technical material (including the definition of the map f'), we will record the proof of Theorem 2.3.1 in Paragraph 2.3.25 below.

2.3.3. For a subset T of a finite set S write

$$\varphi_S(T) := \frac{\#T}{\#S}.$$

Let k be a finite field with q elements. Let P_1 and P_2 be projective spaces over k of dimension $n > 3$ and let

$$f : P_1 \rightarrow P_2$$

be an injection of sets. Let \mathcal{L}_{P_i} be the set of lines in P_i . Since the cardinality of P_i is

$$\frac{q^{n+1} - 1}{q - 1},$$

the cardinality of \mathcal{L}_{P_i} is equal to

$$\frac{(q^{n+1} - 1)(q^{n+1} - q)}{q(q+1)(q-1)^2}.$$

Let $\mathcal{L}_{P_1}^f \subset \mathcal{L}_{P_1}$ be the subset of lines $L \subset P_1$ for which $f(L)$ is a line in P_2 . We make the following assumption.

Assumption 2.3.4. For a given $\epsilon > 0$, we have

$$\wp_{\mathcal{L}_{P_1}}(\mathcal{L}_{P_1}^f) \geq 1 - \epsilon.$$

Under the conditions of Assumption 2.3.4, we will explain how to construct a new map

$$f' : P_1 \rightarrow P_2$$

that agrees with f on a large proportion of points. This construction will yield a linear map agreeing with f at most points by applying the usual fundamental theorem of projective geometry to f' , giving us the desired approximate linearization.

2.3.5. The construction of f' follows the recipe described in Remark 2.1.8: Starting with $x \in P_1$ choose two general lines L_1 and L_2 through x . Let $y_1, y_2 \in L_1 - \{x\}$ and $y_3, y_4 \in L_2 - \{x\}$ be randomly chosen points. Let M_1 (resp. M_2) be the line in P_2 through $f(y_1)$ and $f(y_2)$ (resp. $f(y_3)$ and $f(y_4)$). Then we will argue that M_1 and M_2 intersect in a unique point z and define $f'(x) := z$.

To make this precise, let us begin with some calculations. For two points $y_1, y_2 \in P_1$ we can consider the linear span $Sp(y_1, y_2) \subset P_1$, which is either a line (if the points are distinct) or a point. Let $P_1^{2,f} \subset P_1^2$ be the subset of pairs of distinct points y_1, y_2 for which $Sp(y_1, y_2) \in \mathcal{L}_{P_1}^f$.

Lemma 2.3.6.

$$\wp_{P_1^2}(P_1^{2,f}) \geq 1 - \epsilon - \frac{q-1}{q^{n+1}-1}.$$

Proof. We have a surjective map

$$(P_1^2 - \Delta) \rightarrow \mathcal{L}_{P_1}, \quad (y_1, y_2) \mapsto Sp(y_1, y_2),$$

which has fibers of cardinality $q(q+1)$. Here $\Delta \subset P_1^2$ denotes the diagonal. Therefore the number of pairs $(y_1, y_2) \in P_1^2 - \Delta$ for which $f(Sp(y_1, y_2))$ is not a line is at most

$$\epsilon \cdot \#(P_1^2 - \Delta).$$

Therefore

$$\wp_{P_1^2}(P_1^{2,f}) \geq 1 - \epsilon \cdot \frac{\#(P_1^2 - \Delta)}{\#P_1^2} - \frac{\#\Delta}{\#P_1^2}.$$

Since

$$\frac{\#(P_1^2 - \Delta)}{\#P_1^2} \leq 1$$

and

$$\#\Delta = \frac{q^{n+1}-1}{q-1}, \quad \#P_1^2 = \left(\frac{q^{n+1}-1}{q-1}\right)^2$$

we get the result. □

For a fixed point $x \in P_1$, there is a variant of Lemma 2.3.6 where we only consider pairs of points not equal to x . Set

$$(P_1 - \{x\})^{2,f} := (P_1 - \{x\})^2 \cap P_1^{2,f} \subset (P_1 - \{x\})^2.$$

Lemma 2.3.7. *We have*

$$\wp_{(P_1 - \{x\})^2}((P_1 - \{x\})^{2,f}) \geq 1 - \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right) \cdot \left(\frac{q^{n+1}-q}{q^{n+1}-1} \right)^{-2}.$$

Proof. Using Lemma 2.3.6 we have

$$\#((P_1 - \{x\})^2 - (P_1 - \{x\})^{2,f}) \leq \#(P_1^2 - P_1^{2,f}) \leq \#(P_1^2) \cdot \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right).$$

Therefore

$$\wp_{(P_1 - \{x\})^2}((P_1 - \{x\})^{2,f}) \geq 1 - \frac{\#(P_1^2)}{\#((P_1 - \{x\})^2)} \cdot \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right).$$

Now

$$\frac{\#(P_1^2)}{\#((P_1 - \{x\})^2)} = \left(\frac{q^{n+1}-q}{q^{n+1}-1} \right)^{-2}.$$

From this the result follows. \square

Fix a point $x \in P_1$ and let \mathcal{L}_x denote the set of lines through x . Let $\mathcal{L}_x^{(2)}$ denote the set of triples (L, y_1, y_2) , where $L \in \mathcal{L}_x$ and $y_1, y_2 \in L - \{x\}$ are distinct points. For

$$(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

let

$$\pi_{ij} : (\mathcal{L}_x^{(2)})^2 \rightarrow (P_1 - \{x\})^2$$

be the map given by

$$((L_1, y_1, y_2), (L_2, y_3, y_4)) \mapsto (y_i, y_j).$$

Let

$$(\mathcal{L}_x^{(2)})^{2,(i,j)\text{-good}} \subset (\mathcal{L}_x^{(2)})^2$$

denote the subset of data $((L_1, y_1, y_2), (L_2, y_3, y_4))$ for which y_i and y_j are distinct and span a line in $\mathcal{L}_{P_1}^f$.

Lemma 2.3.8.

$$\wp_{(\mathcal{L}_x^{(2)})^2}((\mathcal{L}_x^{(2)})^{2,(i,j)\text{-good}}) \geq 1 - \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right) \cdot \left(\frac{q^{n+1}-q}{q^{n+1}-1} \right)^{-2}.$$

Proof. Indeed this follows from Lemma 2.3.7 and the observation that the map π_{ij} is surjective with fibers of equal cardinality $(q-1)^2$. \square

Let $\mathcal{S} \subset (\mathcal{L}_x^{(2)})^2$ denote the subset of data $((L_1, y_1, y_2), (L_2, y_3, y_4))$ such that

$$Sp(y_1, y_3), Sp(y_2, y_4) \in \mathcal{L}_x^f$$

and $Sp(f(y_1), f(y_2))$ and $Sp(f(y_3), f(y_4))$ have a unique intersection point.

Lemma 2.3.9.

$$\wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}) \geq 1 - A(q, n, \epsilon).$$

Proof. Two lines in P_i are coplanar if and only if they intersect in exactly one point. From this it follows that for data

$$(2.3.9.1) \quad ((L_1, y_1, y_2), (L_2, y_3, y_4)) \in (\mathcal{L}_x^{(2)})^{2,(1,3)\text{-good}} \cap (\mathcal{L}_x^{(2)})^{2,(2,4)\text{-good}}$$

the points $(f(y_1), f(y_2), f(y_3), f(y_4))$ are coplanar. Indeed because the points (y_1, y_2, y_3, y_4) are coplanar, the lines $Sp(y_1, y_3)$ and $Sp(y_2, y_4)$ intersect in a unique point from which it follows that the lines

$$Sp(f(y_1), f(y_3)) = f(Sp(y_1, y_3)), \quad Sp(f(y_2), f(y_4)) = f(Sp(y_2, y_4))$$

are coplanar (since they intersect in a unique point).

Let $\mathcal{S}^c \subset (\mathcal{L}_x^{(2)})^{2,(1,3)\text{-good}} \cap (\mathcal{L}_x^{(2)})^{2,(2,4)\text{-good}}$ be the subset of the collections of data (2.3.9.1) for which

$$Sp(f(y_1), f(y_2)) = Sp(f(y_3), f(y_4)).$$

From this discussion we then have

$$\wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}) \geq 1 - 2 \left(\epsilon + \frac{q-1}{q^{n+1}-1} \right) \cdot \left(\frac{q^{n+1}-q}{q^{n+1}-1} \right)^{-2} - \wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}^c).$$

It therefore suffices to show that

$$(2.3.9.2) \quad \wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}^c) \leq \frac{(q-1)^2}{q^{n+1}-q}.$$

The set \mathcal{S}^c is contained in the set of collections of data (2.3.9.1) for which $f(y_3)$ and $f(y_4)$ are each points of the line $Sp(f(y_1), f(y_2))$. Since f is a injection the cardinality of this set is less than or equal to

$$(\#\mathcal{L}_x^{(2)}) \cdot (q-1)^2,$$

and so we obtain the inequality (2.3.9.2). □

2.3.10. For two collections

$$((L_1, y_1, y_2), (L_2, y_3, y_4)), ((L'_1, y'_1, y'_2), (L'_2, y'_3, y'_4)) \in \mathcal{S},$$

we then get two intersection points

$$z := Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)), \quad z' := Sp(f(y'_1), f(y'_2)) \cap Sp(f(y'_3), f(y'_4)).$$

We are interested in the probability that $z = z'$.

For $z \in P_2$ let

$$\mathcal{S}_z \subset \mathcal{S}$$

be the subset of pairs

$$((L_1, y_1, y_2), (L_2, y_3, y_4)) \in \mathcal{S}$$

for which

$$Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)) = z.$$

Lemma 2.3.11. *There exists $z \in P_2$ such that*

$$\wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}_z) \geq 1 - 2A(q, n, \epsilon) - 2 \frac{q^2(q+1)^2}{q^{n+1}-q}.$$

Proof. For $((L_1, y_1, y_2), (L_2, y_3, y_4)) \in \mathcal{S}$ with

$$Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)) = z$$

the number of triples $(L_3, y_5, y_6) \in \mathcal{L}_x^{(2)}$ for which the intersections

$$(2.3.11.1) \quad Sp(f(y_5), f(y_6)) \cap Sp(f(y_1), f(y_2)), \quad Sp(f(y_5), f(y_6)) \cap Sp(f(y_3), f(y_4))$$

consist of single points not equal to z can be bounded as follows. For each $a_1 \in Sp(f(y_1), f(y_2))$ and $a_2 \in Sp(f(y_3), f(y_4))$ there exists a unique line $L_{a_1, a_2} \subset P_2$ through a_1 and a_2 , and there are $(q+1)q$ pairs of ordered points (w_5, w_6) on this line. Now if $(L_3, y_5, y_6) \in \mathcal{L}_x^{(2)}$ is such that the intersections (2.3.11.1) consist of single points not equal to z , then we must have $(y_5, y_6) = (f^{-1}(w_5), f^{-1}(w_6))$ for some such pair (w_5, w_6) (with a_1 and a_2 the two respective intersections). Since L is determined by (y_5, y_6) this shows that the number of such triples (L_3, y_5, y_6) is bounded by $q(q+1)$ for a given (a_1, a_2) .

Let $\mathcal{T} \subset (\mathcal{L}_x^{(2)})^4$ be the set of collections of data

$$(2.3.11.2) \quad \{(L_1, y_1, y_2), (L_2, y_3, y_4), (L'_1, y'_1, y'_2), (L'_2, y'_3, y'_4)\} \in (\mathcal{L}_x^{(2)})^4$$

for which any pair of elements in this set is in \mathcal{S} . We then get from the preceding discussion that if $\mathcal{T}_{\text{bad}} \subset \mathcal{T}$ denotes the subset of elements for which

$$Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)) \neq Sp(f(y'_1), f(y'_2)) \cap Sp(f(y'_3), f(y'_4))$$

then

$$\#\mathcal{T}_{\text{bad}} \leq 2|\mathcal{S}| \cdot q^2(q+1)^2 \cdot \#(\mathcal{L}_x^{(2)}).$$

Setting

$$\mathcal{T}_{\text{good}} := \mathcal{T} - \mathcal{T}_{\text{bad}}$$

we find that

$$\wp_{(\mathcal{L}_x^{(2)})^4}(\mathcal{T}_{\text{good}}) \geq \wp_{(\mathcal{L}_x^{(2)})^4}(\mathcal{S}^2) - 2 \frac{q^2(q+1)^2}{\#\mathcal{L}_x^{(2)}}.$$

Now by Lemma 2.3.9 we have

$$\wp_{(\mathcal{L}_x^{(2)})^4}(\mathcal{S}^2) \geq 1 - 2A(q, n, \epsilon),$$

and since

$$\#\mathcal{L}_x^{(2)} = q^{n+1} - q$$

we find that

$$(2.3.11.3) \quad \wp_{(\mathcal{L}_x^{(2)})^4}(\mathcal{T}_{\text{good}}) \geq 1 - 2A(q, n, \epsilon) - 2 \frac{q^2(q+1)^2}{q^{n+1} - q}.$$

Let

$$t : (\mathcal{L}_x^{(2)})^4 \rightarrow (\mathcal{L}_x^{(2)})^2$$

be the projection sending (2.3.9.2) to the pair

$$((L_1, y_1, y_2), (L_2, y_3, y_4)).$$

From the inequality (2.3.11.3) we then find that there exists

$$((L_1^g, y_1^g, y_2^g), (L_2^g, y_3^g, y_4^g)) \in \mathcal{S}$$

such that

$$\wp_{(\mathcal{L}_x)^2}(\mathcal{T}_{\text{good}} \cap t^{-1}(((L_1^g, y_1^g, y_2^g), (L_2^g, y_3^g, y_4^g)))) \geq 1 - 2A(q, n, \epsilon) - 2\frac{q^2(q+1)^2}{q^{n+1} - q}.$$

Let z denote the point

$$z := Sp(f(y_1^g), f(y_2^g)) \cap Sp(f(y_3^g), f(y_4^g)).$$

Then

$$\mathcal{T}_{\text{good}} \cap t^{-1}(((L_1^g, y_1^g, y_2^g), (L_2^g, y_3^g, y_4^g))) \subset \mathcal{S}_z$$

and therefore we have found z as in the lemma. \square

Corollary 2.3.12. *If*

$$(2.3.12.1) \quad A(q, n, \epsilon) + 2\frac{q^2(q+1)^2}{q^{n+1} - q} < \frac{1}{2}$$

then there exists a unique point $z \in P_2$ such that

$$\wp_{(\mathcal{L}_x^{(2)})^2}(\mathcal{S}_z) \geq \frac{1}{2}.$$

Proof. Indeed this follows from Lemma 2.3.11 and the fact that the \mathcal{S}_z 's are disjoint. \square

Assumption 2.3.1. Assume for the rest of the discussion that the inequality (2.3.12.1) holds.

2.3.13. We define a map

$$f' : P_1 \rightarrow P_2$$

by sending $x \in P_1$ to the point $z \in P_2$ given by Corollary 2.3.12.

Let $P_1^{f=f'} \subset P_1$ be the set of points x for which $f(x) = f'(x)$.

Lemma 2.3.14.

$$\wp_{P_1}(P_1^{f=f'}) \geq 1 - 2\epsilon - 2A(q, n, \epsilon) - 2\frac{q^2(q+1)^2}{q^{n+1} - q}.$$

Proof. Let $(\mathcal{L}_-^{(2)})^2$ denote the set of collections

$$(2.3.14.1) \quad (x, ((L_1, y_1, y_2), (L_2, y_3, y_4))),$$

where $x \in P_1$ and $((L_1, y_1, y_2), (L_2, y_3, y_4)) \in (\mathcal{L}_x^{(2)})^2$. We have two maps

$$F, F' : (\mathcal{L}_-^{(2)})^2 \rightarrow P_2$$

given by

$$F((x, ((L_1, y_1, y_2), (L_2, y_3, y_4)))) = x$$

and

$$F'((x, ((L_1, y_1, y_2), (L_2, y_3, y_4)))) = f'(x),$$

and it suffices to calculate the proportion of elements for which $F = F'$ since the map

$$(\mathcal{L}_x^{(2)})^2 \rightarrow P_1$$

given by x is surjective with constant fiber size.

Let $(\mathcal{L}_-^{(2)})_{F\text{-good}}^2 \subset (\mathcal{L}_-^{(2)})^2$ denote the subset of collections for which f takes the lines

$$Sp(y_1, y_2), \quad Sp(y_3, y_4)$$

to lines in P_2 . For collections in $(\mathcal{L}_-^{(2)})_{F\text{-good}}^2$ we then have

$$f(x) = Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)).$$

Define $(\mathcal{L}_-^{(2)})_{F'\text{-good}}^2 \subset (\mathcal{L}_-^{(2)})^2$ to be the subset of collections for which

$$((L_1, y_1, y_2), (L_2, y_3, y_4)) \in \mathcal{S}_{f'(x)} \subset (\mathcal{L}_x^{(2)})^2.$$

We have

$$\wp_{(\mathcal{L}_-^{(2)})^2}((\mathcal{L}_-^{(2)})_{F\text{-good}}^2) \geq 1 - 2\epsilon.$$

Indeed for (i, j) equal to $(1, 2)$ or $(3, 4)$ the map

$$(\mathcal{L}_-^{(2)})^2 \rightarrow \mathcal{L}_{P_1}$$

is surjective with constant fiber size, and therefore the proportion of collections (2.3.14.1) for which f does not take the line $Sp(y_i, y_j)$ to a line in P_2 is less than or equal to ϵ . Therefore the proportion of collections (2.3.14.1) for which one of $f(Sp(y_1, y_2))$ and $f(Sp(y_3, y_4))$ is not a line is less than or equal to 2ϵ .

For a collection (2.3.14.1) in $(\mathcal{L}_-^{(2)})_{F\text{-good}}^2$ we have

$$f'(x) = Sp(f(y_1), f(y_2)) \cap Sp(f(y_3), f(y_4)).$$

Now by Lemma 2.3.11 we have

$$\wp_{(\mathcal{L}_-^{(2)})^2}((\mathcal{L}_-^{(2)})_{F\text{-good}}^2) \geq 1 - 2A(q, n, \epsilon) - 2\frac{q^2(q+1)^2}{(q^{n+1} - q)}.$$

To get the lemma note that $F = F'$ on

$$(\mathcal{L}_-^{(2)})_{F\text{-good}}^2 \cap (\mathcal{L}_-^{(2)})_{F'\text{-good}}^2$$

and by the preceding observations we have

$$\wp_{(\mathcal{L}_-^{(2)})^2}((\mathcal{L}_-^{(2)})_{F\text{-good}}^2 \cap (\mathcal{L}_-^{(2)})_{F'\text{-good}}^2) \geq 1 - 2\epsilon - 2A(q, n, \epsilon) - 2\frac{q^2(q+1)^2}{(q^{n+1} - q)}.$$

□

2.3.15. Fix a point $x \in P_1$. Let us calculate a lower bound for the proportion of elements $(L, y_1, y_2) \in \mathcal{L}_x^{(2)}$ for which the following conditions hold:

- (i) $f(y_1) = f'(y_1)$ and $f(y_2) = f'(y_2)$.
- (ii) $(L, y_1, y_2) \in \mathcal{L}_x^{(2)}$ is in the image of the projection map

$$\chi : \mathcal{S}_{f'(x)} \rightarrow \mathcal{L}_x^{(2)}$$

sending $((L_1, y_1, y_2), (L_2, y_3, y_4))$ to (L_1, y_1, y_2) .

For $j = 1, 2$ the map

$$\mathcal{L}_x^{(2)} \rightarrow P_1 - \{x\}, \quad (L, y_1, y_2) \mapsto y_j$$

is surjective with constant fiber size $q - 1$. It follows from this and Lemma 2.3.14 that the number of (L, y_1, y_2) for which (i) fails is bounded above by

$$2(q - 1) \cdot \frac{\#P_1}{\#P_1 - 1} \cdot \left(2\epsilon + 2A(q, n, \epsilon) + 2\frac{q^2(q + 1)^2}{q^{n+1} - q} \right).$$

As for condition (ii), note that the complement of the image of χ is at most of size

$$\#\mathcal{L}_x^{(2)} - \frac{\#\mathcal{D}_{f'(x)}}{\#\mathcal{L}_x^{(2)}}.$$

Therefore using Lemma 2.3.11 we find that the proportion of elements of $\mathcal{L}_x^{(2)}$ for which (i) or (ii) fails is bounded above by $B(q, n, \epsilon)$.

Now if $(L, y_1, y_2) \in \mathcal{L}_x^{(2)}$ satisfies (i) and (ii), then it follows that $f'(x), f'(y_1), f'(y_2) \in P_2$ are collinear. We summarize this in the following lemma:

Lemma 2.3.16. *We have*

$$\wp_{\mathcal{L}_x^{(2)}} \left(\{(L, y_1, y_2) \in \mathcal{L}_x^{(2)} \mid (f'(x), f'(y_1), f'(y_2)) \text{ are collinear}\} \right) \geq 1 - B(q, n, \epsilon).$$

Proof. This follows from the preceding discussion. \square

2.3.17. We will use Lemma 2.3.16 to show that f' takes lines to lines and that f' is injective. For this we will use Desargues's theorem, which is a consequence of Pappus's axiom, and the notion of Desargues configurations.

Recall that a *Desargues configuration* is a collection of 10 points and 10 lines such that any line contains exactly three of the points and exactly three lines pass through each point.

Desargues theorem can be stated as follows. Consider two collections of three points $\{A, B, C\}$ and $\{D, E, F\}$, usually thought of as the vertices of two triangles, and consider the 9 lines

$$\{AB, AC, BC, DE, DF, EF, AD, BE, CF\}.$$

Theorem 2.3.18 (Desargues). *If the three lines $AD, BE,$ and CF meet in a common point G then the three intersection points*

$$H := AB \cap DE, \quad I := AC \cap DF, \quad J := BC \cap EF,$$

are collinear, and conversely if these three points are collinear then the lines $AD, BE,$ and CF meet at a common point.

In other words, the ten points and ten lines obtained in this way form a Desargues configuration.

2.3.19. To show that f' takes lines to lines, it therefore suffices to show that for any three collinear points (x, y, t) there exists a Desargues configuration as above with $(H, I, J) = (x, y, t)$ such that f' takes all the lines other than $Sp(x, y)$ to lines in P_2 . For then, by Desargues's theorem, it follows that $(f'(x), f'(y), f'(t))$ are collinear. We will produce such a Desargues configuration using basic linear algebra. We fix the collinear points $\{x, y, t\}$ in what follows.

Notation 2.3.20. Let V_1 be an F -vector space with $PV_1 = P_1$, and choose vectors $a, b \in V_1$ such that (x, y, t) is given by the three elements $(a, b, a - b) \in V_1$.

Construction 2.3.21. For $c, d \in V_1$, consider the ordered set of five elements $\{0, a, b, c, d\}$. Let $\mathcal{P}(c, d)$ denote the set of points of P_1 given by the differences of two elements

$$\mathcal{P}(c, d) := \{[a], [b], [c], [d], [b - a], [c - a], [d - a], [c - b], [d - b], [c - d]\},$$

and let $\mathcal{M}(c, d)$ denote the set of lines obtained by taking for each subset of three elements $T \subset \{0, a, b, c, d\}$ the linear span L_T of differences of elements of T .

Lemma 2.3.22. *As long as the set of four elements $\{a, b, c, d\}$ are linearly independent the ten points and ten lines $(\mathcal{P}(c, d), \mathcal{M}(c, d))$ form a Desargues configuration.*

Proof. The proof is routine linear algebra. □

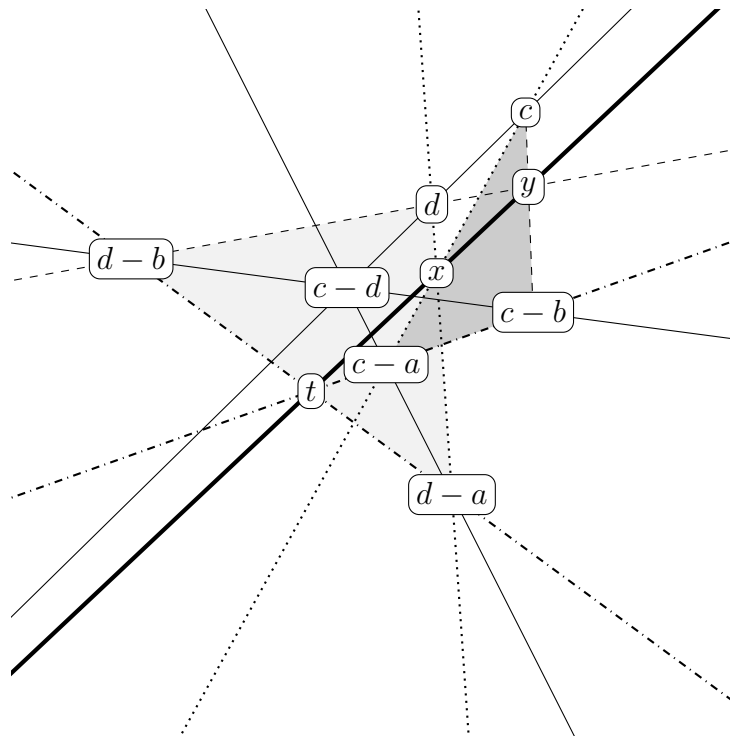


FIGURE 2.3.1. Construction Construction 2.3.21

Fig. 2.3.1 shows a typical configuration generated by Construction 2.3.21 (on a true set of randomly generated data). The bold line shows the collinear points x, y , and t , together with the auxiliary points given by the choices of c and d . Some of the lines naturally come in pairs, corresponding to the construction of the map f' in Paragraph 2.3.13. (For example, the dotted line connecting x to d and $d - a$ and the dotted line connecting x to c and $c - a$ serve to define $f'(x)$, under the assumption that those two lines are mapped to lines under f .) The remaining solid lines complete the Desargues configuration. The two perspective triangles are shaded in gray. The center of perspectivity lies at $c - d$, and the axis of perspectivity is the line spanned by x, y , and t .

Notation 2.3.23. Let $W \subset V_1^{\times 2}$ be the subset of pairs (c, d) such that the following conditions hold:

- (i) The set of ten lines and ten points $(\mathcal{P}(c, d), \mathcal{M}(c, d))$ of Construction 2.3.21 is a Desargues configuration.
- (ii) For all $P \in \mathcal{P}(c, d)$ not in $\{x, y, t\}$ we have $f(P) = f'(P)$.
- (iii) The map f' takes every line in $\mathcal{M}(c, d) \setminus \{Sp(x, y)\}$ to a line in P_2 .

We can find a lower bound for the size of W as follows. Recall the function B from (2.3.0.2).

Proposition 2.3.24. *We have*

$$\wp_{V_1^{\times 2}}(W) \geq 1 - 9B(q, n, \epsilon) - q^{-n+3}.$$

In particular, if

$$9B(q, n, \epsilon) + q^{-n+3} < 1$$

then $W \neq \emptyset$.

Proof. Let $X \subset V_1^{\times 2}$ be the subset of pairs (c, d) for which $\{a, b, c, d\}$ are linearly independent. Letting X^c denotes the complement of X in $V_1^{\times 2}$, note that

$$\#X^c \leq q^{n+4}.$$

For a subset $T \subset \{0, 1, 2, 3, 4\}$ of size 3 not equal to $\{0, 1, 2\}$ we have a map

$$\pi_T : X \rightarrow \mathcal{L}^{(2)}$$

sending (c, d) to the linear span of differences of elements of $T \subset \{0, 1, 2, 3, 4\} \simeq \{0, a, b, c, d\}$, with the two points defined by the differences of the first and last element, and second and last element of T .

If T meets $\{0, 1, 2\}$ in two elements, so that one of the points $\{x, y, t\}$ lies on $\pi_T(c, d)$ for any (c, d) , then π_T surjects onto $\mathcal{L}_z^{(2)}$ for some $z \in \{x, y, t\}$ with constant fiber size, and if T meets $\{0, 1, 2\}$ in one element then π_T is surjective onto $\mathcal{L}^{(2)}$ with constant fiber size.

Combining this with Lemma 2.3.14 and the discussion in Paragraph 2.3.15 it follows that if $X^T \subset X$ is the set of pairs (c, d) such that the linear span of differences of elements of T is taken to a line in P_2 under f' and such that $f' = f$ on the points of this line corresponding to elements of $\mathcal{P} - \{x, y, t\}$, then

$$\wp_X(X^T) \geq 1 - B(q, n, \epsilon).$$

Note that

$$\cap_T X^T = W,$$

and since there are nine choices of T we find that

$$\wp_X(W) \geq 1 - 9B(q, n, \epsilon).$$

Combining this with our estimate for X^c we find that

$$\wp_{V_1^{\times 2}}(W) \geq 1 - 9B(q, n, \epsilon) - q^{-n+3},$$

as desired. □

2.3.25. We are now ready to give the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. We let f' be the map defined in Paragraph 2.3.13. We refer in this proof to the diagram in Fig. 2.3.1.

Assuming the inequality of Proposition 2.3.24, we can choose $(c, d) \in W$, and let $(\mathcal{P}, \mathcal{M}) = (\mathcal{P}(c, d), \mathcal{M}(c, d))$ be the resulting Desargues configuration. We have that all the lines in \mathcal{M} , except possibly for (x, y, t) , are taken to lines in P_2 under f' . Thus, in Fig. 2.3.1, the dotted, dashed, dot-dashed, and non-bold solid lines are all taken to lines under f' . On the other hand, the images of the dotted lines intersect at $f'(x)$, the images of the dashed lines intersect at $f'(y)$, and the images of the dot-dashed lines intersect at $f'(t)$. By Desargues theorem, $f'(x)$, $f'(y)$, and $f'(t)$ are collinear and distinct, lying on the axis of perspectivity for the image Desargues configuration. Note that this also implies that f' is injective, \square

3. DIVISORIAL STRUCTURES AND DEFINABLE LINEAR SYSTEMS

3.1. Divisorial structures. In this section we introduce the key structure that will ultimately be the subject of our main reconstruction theorem. Recall that, given a Zariski topological space Z , an *effective divisor* is a formal finite sum $\sum a_i x_i$, where each $x_i \in X$ is a point of codimension 1 and each a_i is positive. We denote the set of effective divisors on Z by $\text{Eff}(Z)$. When X is a scheme, we will write $\text{Eff}(X)$ for $\text{Eff}(|X|)$.

Definition 3.1.1. An *absolute variety* is a scheme X such that the following conditions hold:

- (i) X is integral and $\kappa_X := \Gamma(X, \mathcal{O}_X)$ is a field.
- (ii) The canonical morphism $X \rightarrow \text{Spec}(\kappa_X)$ is separated, of finite type, and has integral geometric fibers.

An absolute variety is *polarizable* if X admits an ample invertible sheaf.

Remark 3.1.2. In what follows we refer to κ_X as the *constant field* of X .

Definition 3.1.3. A normal separated k -scheme X is *divisorially proper over k* if for any reflexive sheaf L of rank 1 we have that $\Gamma(X, L)$ is finite-dimensional over k .

Lemma 3.1.4. *If a k -scheme X is normal, separated, and divisorially proper over k and $U \subset X$ is an open subscheme such that $\text{codim}(X \setminus U \subset X) \geq 2$ at every point, then U is also divisorially proper over k .*

Proof. Any reflexive sheaf L of rank 1 on U is the restriction of a reflexive sheaf L' of rank 1 on X , and Krull's theorem tells us that the restriction map

$$\Gamma(X, L') \rightarrow \Gamma(U, L)$$

is an isomorphism of k -vector spaces. \square

Definition 3.1.5. A absolute variety X is *definable* if it is normal and divisorially proper over $\Gamma(X, \mathcal{O}_X)$.

Write \mathcal{V}_{ar} for the category whose objects are absolute varieties and whose morphisms are open immersions $f : X \rightarrow Y$ such that $Y \setminus f(X)$ has codimension at least 2 in Y at every point. We will write $\mathcal{D}ef \subset \mathcal{V}_{ar}$ for the full subcategory of definable schemes.

Definition 3.1.6. A *divisorial structure* is a pair (Z, Λ) with Z a Zariski topological space and Λ a congruence relation on the monoid $\text{Eff}(Z)$.

Definition 3.1.7 (Restriction of a divisorial structure). Suppose $t := (Z, \Lambda)$ is a divisorial structure. Given an open subset $U \subset Z$, the *restriction of t to U* , denoted $t|_U$, is the divisorial structure $(U, \Lambda_{\text{Eff}(U)})$, where $\Lambda_{\text{Eff}(U)}$ is the induced relation on the quotient monoid $\text{Eff}(X) \rightarrow \text{Eff}(U)$.

In other words, if we let $\text{Eff}(X) \rightarrow Q$ denote the quotient by Λ , we define the congruence relation on $\text{Eff}(U)$ by forming the pushout

$$\begin{array}{ccc} \text{Eff}(X) & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \text{Eff}(U) & \longrightarrow & Q_U \end{array}$$

in the category of commutative integral monoids.

Alternatively, recall that the condition that an equivalence relation $\Lambda \subset \text{Eff}(Z) \times \text{Eff}(Z)$ is a congruence relation is equivalent to the condition that Λ is a submonoid. The congruence relation on $\text{Eff}(U)$ induced by Λ is simply the image of Λ under the surjective map

$$\text{Eff}(Z) \times \text{Eff}(Z) \rightarrow \text{Eff}(U) \times \text{Eff}(U).$$

Definition 3.1.8 (Morphisms of divisorial structures). A *morphism of divisorial structures*

$$(Z, \Lambda) \rightarrow (Z', \Lambda')$$

is an open immersion of topological spaces $f : Z \rightarrow Z'$ such that

$$\text{Eff}(f) : \text{Eff}(Z) \rightarrow \text{Eff}(Z')$$

is a bijection and

$$(\text{Eff}(f) \times \text{Eff}(f))(\Lambda) = \Lambda'.$$

Notation 3.1.9. We will write \mathcal{T} for the category of divisorial structures.

Definition 3.1.10. The *divisorial structure* of an integral scheme X is the pair

$$\tau(X) := (|X|, \Lambda_X),$$

where $|X|$ is the underlying Zariski topological space of X and

$$\Lambda_X \subset \text{Eff}(X) \times \text{Eff}(X)$$

is the rational equivalence relation on effective divisors.

Remark 3.1.11. The divisorial structure of an integral scheme X can be obtained from the data of the triple

$$(|X|, \text{Cl}(X), c : X^{(1)} \rightarrow \text{Cl}(X)).$$

Indeed by the universal property of a free monoid on a set giving the map c is equivalent to giving a map of monoids

$$\text{Eff}(X) \rightarrow \text{Cl}(X),$$

and the congruence relation defined by this map is precisely the equivalence relation given by rational equivalence. Conversely, from the equivalence relation on $\text{Eff}(X)$ we obtain the class group as the group associated to the quotient of $\text{Eff}(X)$ by the congruence relation and the map c is induced by the natural map $X^{(1)} \rightarrow \text{Eff}(X)$.

Formation of the divisorial structures defines a diagram of categories

$$(3.1.11.1) \quad \mathcal{D}ef \subset \mathcal{V}ar \xrightarrow{\tau} \mathcal{T}$$

The main result of this paper is the following.

Theorem 3.1.12. *The functor $\tau|_{\mathcal{D}ef}$ is fully faithful.*

The proof of Theorem 3.1.12 will be given in Section 4 after some preliminary foundational work.

3.2. Some remarks on divisors. In this section we gather a few facts about divisors on normal varieties. Our main purpose is to demonstrate that some basic features of such varieties – such as the maximal factorial open subscheme – can be characterized purely in terms of the divisorial structure.

Fix a field k . For a normal irreducible separated k -scheme X let

$$q : \text{Eff}(X) \rightarrow \overline{\text{Eff}}(X)$$

denote the quotient monoid given by rational equivalence of divisors, so that $\overline{\text{Eff}}(X)$ is the image of $\text{Eff}(X)$ in $\text{Cl}(X)$. Given a divisor D on X , upon identifying $|D|$ with the subset of effective divisors on X that are linearly equivalent to D , we have a set-theoretic equality

$$|D| = q^{-1}(q(D)).$$

In particular, the linear system is defined as a set by the map q .

There is a reflexive sheaf of rank 1 canonically associated to D that we will write $\mathcal{O}(D)$. Members of $|D|$ are in bijection with sections $\mathcal{O} \rightarrow \mathcal{O}(D)$ in the usual way. Recall that D is Cartier if and only if $\mathcal{O}(D)$ is an invertible sheaf on X .

Lemma 3.2.1. *Let $U \subset X$ be an open subscheme. Then the commutative diagram*

$$\begin{array}{ccc} \text{Eff}(X) & \longrightarrow & \overline{\text{Eff}}(X) \\ \downarrow & & \downarrow \\ \text{Eff}(U) & \longrightarrow & \overline{\text{Eff}}(U) \end{array}$$

is a pushout diagram in the category of integral monoids.

Proof. If $E_1, E_2 \in \text{Eff}(U)$ are two classes mapping to the same class in $\overline{\text{Eff}}(U)$ then there exists a rational function $f \in \overline{\text{Eff}}(U)$ such that

$$\text{div}_U(f) = E_1 - E_2$$

in $\text{Div}(U)$. Then $\text{div}_X(f) \in \text{Div}(X)$ maps to $\text{div}_U(f)$ in $\text{Div}(U)$, so if we write $\text{div}_X(f)$ as

$$\tilde{E}_1 - \tilde{E}_2,$$

where the divisors \tilde{E}_i are effective, then $\tilde{E}_i \in \text{Eff}(X)$ are rationally equivalent divisors mapping to the E_i . This shows that the equivalence relation on $\text{Eff}(U)$ given by rational equivalence is the image of equivalence relation on $\text{Eff}(X)$ given by the projection $\text{Eff}(X) \rightarrow \overline{\text{Eff}}(X)$, which implies the lemma. \square

Corollary 3.2.2. *If X is an integral scheme and $U \subset X$ is an open subscheme then the divisorial structure $\tau(U)$ is canonically isomorphic to the restriction $\tau(X)|_U$ (see Definition 3.1.7).*

Proof. The equivalence relation on $\text{Eff}(X)$ is the relation defined by the quotient map $\text{Eff}(X) \rightarrow \overline{\text{Eff}}(X)$. By Lemma 3.2.1, we see that the induced relation on $\tau(X)|_U$ is precisely the relation for $\tau(U)$, giving the desired result. \square

Definition 3.2.3. Given an excellent scheme X , the *Cartier locus* of X is the largest open subscheme $U \subset X$ that is factorial (i.e., such that every Weil divisor on U is Cartier).

Proposition 3.2.4. *Let X be a normal irreducible quasi-compact separated scheme and let $D \subset X$ be a divisor.*

- (1) *If $|D|$ is basepoint free then D is Cartier.*
- (2) *If D is ample then D is \mathbb{Q} -Cartier.*

Proof. Since X is quasi-compact, if D is ample we know that $|nD|$ is basepoint free for some n . Thus it suffices to prove the first statement. Given a point $x \in X$, choose $E \in |D|$ such that $x \notin E$. This gives some section $s : \mathcal{O} \rightarrow \mathcal{O}(D)$. Restricting to the local ring $R = \mathcal{O}_{X,x}$, we see that $s_x : R \rightarrow \mathcal{O}(D)_x$ is an isomorphism in codimension 1 (for otherwise E would be supported at x). Since $\mathcal{O}(D)$ is reflexive, it follows that s_x is an isomorphism, whence $\mathcal{O}(D)$ is invertible in a neighborhood of x . Since this holds at any $x \in X$, we conclude that $\mathcal{O}(D)$ is invertible, as desired. \square

Corollary 3.2.5. *A normal irreducible separated scheme X is factorial if and only if it is covered by open subschemes $U \subset X$ with the property that every divisor class on U is basepoint free.*

Proof. If X is factorial, then any affine open covering has the desired property, since any Cartier divisor on an affine scheme is basepoint free. On the other hand, if X admits such a covering, then we know that every divisor class on X is locally Cartier, whence it is Cartier. \square

Proposition 3.2.6. *If X is a normal k -variety then we can characterize the Cartier locus of X as the union of all open subsets $U \subset X$ such that every divisor class on U is basepoint free.*

Proof. This is an immediately consequence of Corollary 3.2.5. \square

The preceding discussion implies that various properties of a scheme X and its divisors can be read off from the divisorial structure. We summarize this in the following.

Proposition 3.2.7. *Let X be a normal separated quasi-compact and irreducible scheme and let*

$$\tau(X) = (|X|, \Lambda_X)$$

be the associated divisorial structure. Then

- (i) *the property that $D \in \text{Eff}(X)$ has basepoint free linear system $|D|$ depends only on $\tau(X)$;*
- (ii) *the property that X is factorial depends only on $\tau(X)$;*
- (iii) *the Cartier locus of X depends only on $\tau(X)$;*
- (iv) *the condition that a divisor D is ample depends only on $\tau(X)$.*

Proof. Let

$$q : \text{Eff}(X) \rightarrow \overline{\text{Eff}}(X)$$

denote the quotient map defined by Λ_X , so that for $D \in \text{Eff}(X)$ we have $|D| = q^{-1}q(D)$. The condition that $|D|$ is base point free is the statement that for every $x \in |X|$ there exists $E \in |D|$ such that $x \notin E$. Evidently this depends only on $\tau(X)$, proving (i).

Likewise the condition that a divisor D is ample is the statement that the open sets defined by elements of $|nD|$ for $n \geq 0$ give a base for the topology on $|X|$. Again this clearly only depends on $\tau(X)$, proving (iv).

Statement (ii) follows from Corollary 3.2.5 and Lemma 3.2.1, which implies that the divisorial structure $\tau(U)$ for an open subset $U \subset X$ is determined by $|U| \subset |X|$ and $\tau(X)$.

Finally (iii) follows from Proposition 3.2.6. □

3.2.8. Our proof of Theorem 3.1.12 will ultimately rely on reducing to the projective case. For the remainder of this section, we record some results about polarizations that we will need later.

Given a definable scheme X , write $\text{Abpf}(X) \subset \text{Eff}(X)$ for the (possibly empty) submonoid of ample basepoint free effective divisors D and $\text{Abpfi}(X) \subset \text{Abpf}(X)$ for the submonoid of divisors whose associated linear system defines an injective map $\nu_D : X \hookrightarrow |D|^\vee$.

Lemma 3.2.9. *Suppose given two definable schemes X and Y and an isomorphism $\varphi : \tau(X) \rightarrow \tau(Y)$. If X is polarizable and factorial then so is Y and φ induces a commutative diagram of monoids*

$$\begin{array}{ccc} \text{Abpf}(X) & \longrightarrow & \text{Eff}(X) \\ \wr \downarrow & & \downarrow \wr \\ \text{Abpf}(Y) & \longrightarrow & \text{Eff}(Y) \end{array}$$

in which the vertical arrows are isomorphisms. If the constant fields of X and Y are algebraically closed, then the isomorphism $\text{Abpf}(X) \rightarrow \text{Abpf}(Y)$ restricts to an isomorphism $\text{Abpfi}(X) \rightarrow \text{Abpfi}(Y)$ so we obtain a commutative diagram

$$\begin{array}{ccc} \text{Abpfi}(X) & \longrightarrow & \text{Abpf}(X) \\ \wr \downarrow & & \downarrow \wr \\ \text{Abpfi}(Y) & \longrightarrow & \text{Abpf}(Y) \end{array}$$

Proof. Since X is factorial all divisors are Cartier divisors. By Proposition 3.2.7, Y is also factorial and polarizable, and the submonoid Abpf is preserved, as claimed. Finally, when κ_X and κ_Y are algebraically closed, one can tell if ν_D is injective by

seeing if the sets $H_x = \{E \in |D| : x \in E\}$ are distinct for distinct closed points x ; thus, $\text{Abpfi}(X)$, resp. $\text{Abpfi}(Y)$, is determined by $\tau(X)$, resp. $\tau(Y)$. (Note that it is not yet clear if κ_X is determined by $\tau(X)$. This will be discussed in Section 2.1 and Section 4 below.) \square

Definition 3.2.10. Suppose X is a definable scheme. An open subscheme $U \subset X$ will be called *essential* if $\text{codim}(X \setminus U \subset X) \geq 2$, U is factorial, and U is polarizable.

Note that if $U \subset X$ is essential, then the natural restriction map $\text{Eff}(X) \rightarrow \text{Eff}(U)$ is an isomorphism of monoids.

Lemma 3.2.11. *If X is a normal, separated, quasi-compact k -scheme then there is an open subscheme $U \subset X$ such that $\text{codim}(X \setminus U \subset X) \geq 2$ and U is quasi-projective. In particular, any definable scheme X contains an essential open subset $U \subset X$.*

Proof. Working one connected component at a time, we may assume that X is irreducible. By Chow's lemma, there is a proper birational morphism $\pi : \tilde{X} \rightarrow X$ with \tilde{X} quasi-projective. Since X is normal, π is an isomorphism in codimension 1. Thus, \tilde{X} and X have a common open subset U whose complement in X has codimension at least 2, and which is quasi-projective. Passing to the Cartier locus yields the second statement. \square

Lemma 3.2.12. *Suppose X and Y are definable schemes and $\varphi : \tau(X) \rightarrow \tau(Y)$ is an isomorphism of divisorial structures. If $U \subset X$ is an essential open subset then $\varphi(U) \subset Y$ is an essential open subset and there is an induced isomorphism $\tau(U) \xrightarrow{\sim} \tau(\varphi(U))$.*

Proof. First note that since φ induces a homeomorphism $|X| \rightarrow |Y|$, we have that $\text{codim}(X \setminus Y \subset X) = \text{codim}(Y \setminus \varphi(U) \subset Y)$. In particular, if U is definable then so is $\varphi(U)$ by Lemma 3.1.4. By Definition 3.1.7, we have isomorphisms

$$\tau(X)|_U \xrightarrow{\sim} \tau(U)$$

and

$$\tau(Y)|_{\varphi(U)} \xrightarrow{\sim} \tau(\varphi(U)).$$

On the other hand, φ induces an isomorphism $\tau(X)|_U \xrightarrow{\sim} \tau(Y)|_{\varphi(U)}$. The result thus follows from Lemma 3.2.9. \square

3.3. Definable subspaces in linear systems. Fix a definable absolute variety X with infinite constant field. Let $P := |D|$ be the linear system associated to an effective divisor D .

Definition 3.3.1. A subspace $V \subset P$ is *definable* if there is a subset $Z \subset X$ such that

$$V = V(Z) := \{E \in P \mid Z \subset E\}.$$

Remark 3.3.2. If $Z \subset X$ is a subset and $Z' \subset X$ is the closure of Z then $V(Z) = V(Z')$. When considering definable subspaces it therefore suffices to consider subspaces defined by closed subsets.

Remark 3.3.3. Note that $V(Z)$ is the projective space associated to the kernel of the restriction map

$$H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Z_{\text{red}}, \mathcal{O}_X(D)|_{Z_{\text{red}}}),$$

where we write $Z_{\text{red}} \subset X$ for the reduced subscheme associated to the subspace $Z \subset |X|$.

3.3.4. Algebraically closed constant fields. When the constant field κ_X is algebraically closed, the structure of definable subspaces is significantly simpler than in the general case.

Lemma 3.3.5. *Assume the constant field κ_X is algebraically closed. Suppose $V = V(Z)$ is a non-empty definable subset of a linear system P on X . Then there is an ascending chain of closed subsets*

$$Z = Z_1 \subsetneq \cdots \subsetneq Z_n$$

such that the induced chain

$$V(Z) = V(Z_1) \supsetneq \cdots \supsetneq V(Z_n)$$

is a full flag of linear subspaces ending in a point.

Proof. By induction, it suffices to produce $Z_2 \supsetneq Z_1 = Z$ such that $V(Z_2) \subsetneq V(Z_1)$ has codimension 1.

For this note that if $x \in X$ is a closed point then either

$$V(Z \cup \{x\}) = V(Z)$$

or $V(Z \cup \{x\})$ has codimension 1 in $V(Z)$ since κ_X is algebraically closed. Furthermore, equality happens if and only if $x \in E$ for all $E \in V(Z)$. It therefore suffices to observe that there exists a point $x \in X$ which does not lie in every element of $V(Z)$. This follows from noting that if \mathcal{L} is an invertible sheaf on X and $W \subset \Gamma(X, \mathcal{L})$ is a nonzero subspace then there exists a closed point $x \in X$ such that the map $W \rightarrow \mathcal{L}(x)$ is surjective. □

Corollary 3.3.6. *The dimension of P is equal to one more than the length of a maximal chain of definable subsets.*

Proof. Take $Z = \emptyset$ in Lemma 3.3.5. □

Corollary 3.3.7. *Assume that κ_X is algebraically closed. Given a basepoint free linear system P on X , the definable lines in P are precisely those definable subsets with more than one element that are minimal with respect to inclusions of definable subsets.*

Proof. By Lemma 3.3.5, any definable set of higher dimension contains a definable line. □

Example 3.3.8. The conclusions of Lemma 3.3.5 and Corollary 3.3.7 are false without the assumption that κ_X is algebraically closed. For example, let $X \subset \mathbf{P}_{\mathbf{R}}^2$ be the conic given by

$$V(X^2 + Y^2 + Z^2) \subset \mathbf{P}_{\mathbf{R}}^2,$$

so we have an isomorphism $\sigma : \mathbf{P}_{\mathbf{C}}^1 \simeq X \otimes_{\mathbf{R}} \mathbf{C}$. If \mathcal{L} is an ample invertible sheaf on X then $\sigma^* \mathcal{L}_{\mathbf{C}} \simeq \mathcal{O}_{\mathbf{P}_{\mathbf{C}}^1}(2n)$ for $n > 0$. From this we see that each closed point $x \in X$ imposes a codimension 2 condition on $|\mathcal{L}|$, and since this projective space has dimension $2n$ we conclude that there are no definable lines in any ample linear system on X .

3.3.9. Arbitrary infinite constant fields. Detecting lines is more subtle over arbitrary fields. This is closely related to the counterexamples to Theorem 3.1.12 for curves discussed in Section 4.5.

Lemma 3.3.10. *Let $\ell \subset \mathbf{P}(V)$ be a line corresponding to a two-dimensional subspace $T \subset V$. Let $Z' \subset X$ be the maximal reduced closed subscheme of the intersection of the zero-loci of elements of T . Then ℓ is definable if and only if the dimension of the kernel*

$$K := \ker(\mathrm{H}^0(X, \mathcal{L}) \rightarrow \mathrm{H}^0(Z', \mathcal{L}|_{Z'}))$$

is equal to 2.

Proof. First suppose ℓ is definable, so we can write $\ell = V(Z)$ for some closed subset $Z \subset |X|$, which we view as a subscheme with the reduced structure. Then by definition ℓ is the projective subspace of $\mathbf{P}V$ associated to the kernel of the map

$$\mathrm{H}^0(X, \mathcal{L}) \rightarrow \mathrm{H}^0(Z, \mathcal{L}|_Z),$$

which must therefore equal T . In particular, we have $Z \subset Z'$, which implies that

$$T \subset K \subset \ker(\mathrm{H}^0(X, \mathcal{L}) \rightarrow \mathrm{H}^0(Z, \mathcal{L}|_Z)).$$

It follows that $K = T$, and, in particular, K has dimension 2.

Conversely, if K has dimension 2 then we have $T = K$ and $\ell = V(Z')$. □

Lemma 3.3.11. *Suppose $\mathbf{P}(V)$ is a basepoint free linear system on X . Let $F_1, F_2 \in V$ be two linearly independent vectors with zero loci Z_1 and Z_2 . Assume that*

- (1) Z_1 is reduced;
- (2) the natural map $\mathrm{H}^0(X, \mathcal{O}_X) \rightarrow \mathrm{H}^0(Z_1, \mathcal{O}_{Z_1})$ is an isomorphism;
- (3) the intersection $Z := Z_1 \cap Z_2$ is reduced and does not contain any components of the Z_i .

Then the line in $\mathbf{P}(V)$ spanned by F_1 and F_2 is definable.

Proof. We have short exact sequences

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F_1} \mathcal{L} \longrightarrow \mathcal{L}|_{Z_1} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{Z_1} \xrightarrow{F_2} \mathcal{L}|_{Z_1} \longrightarrow \mathcal{L}|_Z \longrightarrow 0,$$

where the second sequence is exact because Z_1 is reduced and Z does not contain any components of Z_1 . From these sequences we see that if K denotes the kernel of the map

$$\mathrm{H}^0(X, \mathcal{L}) \rightarrow \mathrm{H}^0(Z, \mathcal{L}|_Z)$$

then there is a short exact sequence

$$0 \rightarrow k \cdot F_2 \rightarrow K \rightarrow \mathrm{H}^0(Z_1, \mathcal{O}_{Z_1}).$$

By assumption (2) the right term of this sequence is 1-dimensional, and since K contains the span of F_1 and F_2 it follows that K is 2-dimensional. The result therefore follows from Lemma 3.3.10. □

Proposition 3.3.12. *Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X with associated linear system P . Let $j : X \hookrightarrow \overline{X}$ be the compactification of X provided by the given projective imbedding.*

(i) *Let $V \subset \text{Gr}(1, P)(k)$ be the subset of lines ℓ spanned by elements D and E for which D is geometrically reduced, E is geometrically integral, the intersection $B := E \cap D \subset \overline{X}$ is geometrically reduced and does not contain any components of D or E , and the inclusions*

$$D \cap X \hookrightarrow D, \quad E \cap X \hookrightarrow E, \quad B \cap X \hookrightarrow B$$

are all schematically dense. Then V is the k -points of a dense Zariski open subset of $\text{Gr}(1, P)$ and every element of V is definable.

(ii) *If $D \in |\mathcal{O}_{\overline{X}}(1)| = P$ is a geometrically reduced divisor in \overline{X} for which $D \cap X \subset D$ is dense, then D lies in the sweep of the maximal Zariski open subset of the definable locus in $\text{Gr}(1, P)$.*

Proof. For D and E as in (i) it follows from Lemma 3.3.11 that the span of D and E is a definable line. Furthermore, note that the conditions on D and E are both open conditions. Therefore to prove (i) it suffices to show that V is nonempty, which follows from Bertini's theorem [9, 3.4.10 and 3.4.14].

In fact, given geometrically reduced D with $D \cap X \subset D$ dense, the set of E such that (D, E) satisfy the conditions in (ii) is open and nonempty by [9, 3.4.14]. From this statement (ii) also follows. \square

Corollary 3.3.13. *Let $\sigma : \tau(X) \rightarrow \tau(Y)$ be an isomorphism of divisorial structures such that there are very ample invertible sheaves $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ with σ inducing a bijection*

$$s : |\mathcal{O}_X(1)| \rightarrow |\mathcal{O}_Y(1)|.$$

The map s satisfies the hypotheses of Theorem 2.2.1.

Proof. Indeed, the locus of lines described in Proposition 3.3.12(i) suffices. Note that we know that the linear systems have dimension at least 2 because X and Y have dimension at least 2 and the linear systems are very ample. \square

Example 3.3.14. In general the set of definable lines in $\text{Gr}(1, P)$ is not open. An explicit example is the following.

Consider three k -points $A, B, C \in \mathbf{P}_k^2$, say $A = [0 : 0 : 1]$, $B = [0 : 1 : 0]$, and $C = [1 : 0 : 0]$. For a line $L \subset \mathbf{P}_k^2$ passing through A set

$$T_L := \{F \in H^0(\mathbf{P}_k^2, \mathcal{O}_{\mathbf{P}_k^2}(2)) \mid V(F) \text{ passes through } A, B, C \text{ and is tangent to } L \text{ at } A\}.$$

Concretely if X, Y , and Z are the coordinates on \mathbf{P}_k^2 and L is given by

$$\alpha X + \beta Y = 0,$$

then T_L is given by

$$T_L = \{aXY + b(\alpha X + \beta Y)Z \mid a, b \in k\}.$$

In particular, T_L gives a line ℓ_L in \mathbf{P}_k^2 .

If α and β are nonzero then L does not pass through B and C and the set-theoretic base locus of T_L is equal to $\{A, B, C\}$ and the space of degree two polynomials passing through these three points has dimension 3. Therefore for such L the line ℓ_L is not definable.

However, for L the lines $X = 0$ or $Y = 0$ the line ℓ_L is definable. Indeed in this case the set-theoretic base locus of ℓ_L is given by the union of the line L together with a third point not on the line, from which one sees that T_L is definable.

Letting α and β vary we obtain a 1-parameter family of lines $\mathbf{P}^1 \simeq \Sigma \subset \text{Gr}(1, |\mathcal{O}_{\mathbf{P}^2_k}(2)|)$ whose general member is not definable but with two points giving definable lines. It follows that the definable locus is not open in this case.

Summary 3.3.15. Let us summarize the main consequences of the results in this section. Starting with a projective normal geometrically integral scheme X over an infinite field k we can consider the associated divisorial structure $\tau(X) = (|X|, \Lambda_X)$. From the divisorial structure we can extract several key pieces of information.

- (i) The basepoint free and ample effective divisors and their linear systems are determined by $\tau(X)$. This was discussed in Proposition 3.2.7.
- (ii) If κ_X is algebraically closed field, then for an ample basepoint free linear system \mathbf{P} the set of definable lines in \mathbf{P} is by Corollary 3.3.7 characterized as those definable subsets with more than one element minimal with respect to inclusion. This set depends only on the divisorial structure.
- (ii') If κ_X is an arbitrary infinite field and X has dimension at least 2, then there is an open set of definable lines in any very ample linear system $|\mathcal{O}_X(1)|$, and the bijections $|\mathcal{O}_X(1)| \rightarrow |\mathcal{O}_Y(1)|$ resulting from an isomorphism of divisorial structures send each of these lines to lines (by Theorem 2.2.1).
- (iii) If κ_X is an arbitrary infinite field and $\dim(X) \geq 2$ then for a very ample linear system \mathbf{P} the set of definable lines contains the k -points of a dense open subset of $\text{Gr}(1, P)$, all of whose points are carried to lines by any isomorphism of divisorial structures that carries $\mathcal{O}_X(1)$ to a very ample class.

4. THE UNIVERSAL TORELLI THEOREM

In this section we prove Theorem 3.1.12. Suppose X and Y are definable schemes of dimension at least 2 with infinite constant fields. We need to show that given an isomorphism $\varphi : \tau(X) \rightarrow \tau(Y)$, there is a unique isomorphism of schemes $f : X \rightarrow Y$ such that $\tau(f) = \varphi$.

4.1. Reduction to the quasi-projective case.

Lemma 4.1.1. *If X is a separated Noetherian scheme then for any point $x \in X$ we have that*

$$\overline{\{x\}} = \bigcap \overline{\{y\}},$$

the intersection taken over all points $y \in X$ of codimension at most 1 such that $x \in \overline{\{y\}}$.

Proof. By treating each component of X separately and intersecting the final result, we may assume that X is integral with function field $\kappa(X)$. Since X is separated, the inclusion $x \in X$ is uniquely determined by the inclusion $\mathcal{O}_{X,x} \subset \kappa(X)$. The points y such that $\overline{\{x\}} \subset \overline{\{y\}}$ correspond to those points such that the local ring $\mathcal{O}_{X,y}$ contains $\mathcal{O}_{X,x}$, as subrings of $\kappa(X)$. Letting $I_y \subset \mathcal{O}_{X,x}$ denote the ideal of $\overline{\{y\}}$ in $\mathcal{O}_{X,x}$ (i.e., the intersection $\mathfrak{m}_y \cap \mathcal{O}_{X,x}$ in $\kappa(X)$), we see that we wish to prove that $\mathfrak{m}_x = \sqrt{\sum I_y}$ as $\mathcal{O}_{X,x}$ -modules. Writing $\mathfrak{m}_x = (\alpha_1, \dots, \alpha_n)$, we see that $\{x\} = \cap Z(\alpha_i)$ in $|\text{Spec } \mathcal{O}_{X,x}|$. By

the Krull Hauptidealsatz, each $Z(\alpha_i)$ is a union of codimension 1 closed subschemes that contain x . Taking for $\{y_i\}$ the set of all codimension 1 points that occur among the $Z(\alpha_i)$, we have that $\mathfrak{m}_x = \sqrt{\sum_i I_{y_i}}$, as desired. \square

Lemma 4.1.2. *Suppose $f, g : |X| \rightarrow |Y|$ are homeomorphisms of the underlying spaces of two separated normal Noetherian schemes. Given an open subset $U \subset |X|$ containing all points of codimension 1, if $f|_U = g|_U$ then $f = g$.*

Proof. By Lemma 4.1.1, we can characterize any point $x \in X$ as the unique generic point of an irreducible intersection of closures of codimension ≤ 1 points. But f and g establish the same bijection on the sets of points of codimension ≤ 1 , and, since they are homeomorphisms, therefore the same bijections on the closures of those points. The result follows. \square

Lemma 4.1.3. *Suppose X and Y are normal separated Noetherian schemes, $U \subset X$ and $V \subset Y$ are dense open subschemes with complements of codimension at least 2. Suppose $f : |X| \rightarrow |Y|$ is a homeomorphism of Zariski topological spaces such that $f(U) = V$ and $f|_U$ is the underlying map of an isomorphism $\tilde{f}_U : U \rightarrow V$ of schemes. Then \tilde{f}_U extends to a unique isomorphism of schemes $\tilde{f} : X \rightarrow Y$ whose underlying morphism of topological spaces is f .*

Proof. Let us first show that \tilde{f}_U extends to a morphism of schemes $\tilde{f} : X \rightarrow Y$. If $W_1, W_2 \subset X$ are two open subsets and $\tilde{f}_{W_i} : W_i \rightarrow Y$ ($i = 1, 2$) are morphisms of schemes such that \tilde{f}_{W_i} and \tilde{f}_U agree on $W_i \cap U$, then since Y is separated the morphisms \tilde{f}_{W_1} and \tilde{f}_{W_2} agree on $W_1 \cap W_2$. To extend \tilde{f}_U it therefore suffices to show that \tilde{f}_U extends locally on X . In particular, by covering X by open subsets of the form $\tilde{f}^{-1}(\text{Spec}(A))$ for affines $\text{Spec}(A) \subset Y$, we are reduced to proving the existence of an extension in the case when $Y = \text{Spec}(A)$ is affine. In this case, to give a morphism of schemes $X \rightarrow \text{Spec } A$, it suffices to give a morphism of rings $A \rightarrow \Gamma(X, \mathcal{O}_X)$. By Krull's theorem, $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. Thus, the morphism $\tilde{f}_U : U \rightarrow \text{Spec } A$ extends uniquely to a morphism $\tilde{f} : X \rightarrow \text{Spec } A$, and we get the desired extension \tilde{f} .

Applying the same argument to the inverse of f , and using that X is separated, we see that in fact \tilde{f} is an isomorphism. In particular, its underlying map of topological spaces is a homeomorphism and agrees with f on $|U|$. We conclude by Lemma 4.1.2 that $|\tilde{f}| = f$. \square

4.1.4. From this we get that in order to prove Theorem 3.1.12 it suffices to prove it assuming that X is quasi-projective. Indeed by Lemma 3.2.12, there are essential open subsets $U \subset X$ and $V \subset Y$ such that $V = \varphi(U)$ and τ induces an isomorphism $\tau(U) \rightarrow \tau(V)$. If we know the result in the quasi-projective case then the homeomorphism $|U| \rightarrow |V|$ induced by φ extends to an algebraic isomorphism $f_U : U \rightarrow V$ such that $\tau(f) = \varphi|_U$. By Lemma 4.1.3, f extends uniquely to an isomorphism of schemes $f : X \rightarrow Y$ such that $\tau(f) = \varphi$.

4.2. The quasi-projective case.

4.2.1. For remainder of the proof we assume furthermore that X is quasi-projective. Let $\mathcal{O}_X(1)$ denote a very ample invertible sheaf on X and for $m \geq 1$ let

$$+ : |\mathcal{O}_X(1)|^{\times m} \rightarrow |\mathcal{O}_X(m)|$$

denote the addition map on divisors.

Lemma 4.2.2. *For a general point p of $|\mathcal{O}_X(1)|^{\times m}$ the point $+(p) \in |\mathcal{O}_X(m)|$ lies in the sweep of the maximal Zariski open subset of the set of definable lines in $\text{Gr}(1, |\mathcal{O}_X(1)|)$.*

Proof. Let \bar{X} be the projective closure of X in the embedding given by $\mathcal{O}_X(1)$. Note that \bar{X} is also geometrically integral. Indeed if $j : X \hookrightarrow \bar{X}$ is the inclusion then the map $\mathcal{O}_{\bar{X}} \rightarrow j_*\mathcal{O}_X$ is injective, and remains injective after base field extension. Since X is geometrically integral it follows that \bar{X} is as well.

By Bertini's theorem [9, 3.4.14], for a general choice of $p \in |\mathcal{O}_{\bar{X}}(1)|^{\times m} = |\mathcal{O}_X(1)|^{\times m}$ the point $+(p) \in |\mathcal{O}_{\bar{X}}(1)|$ satisfies the conditions on D in Proposition 3.3.12 (ii) and the result follows. \square

Lemma 4.2.3. *Let X be an integral scheme of dimension d of finite type over k and let $\mathcal{O}_X(1)$ be an ample invertible sheaf on X . Let $z \in X$ be a regular closed point. Then there exists an integer m_0 such that for every $m \geq m_0$ and any $d + 1$ general elements $D_1, \dots, D_{d+1} \in |\mathcal{O}_X(m)|$ containing z we have*

$$\{z\} = |D_1| \cap |D_2| \cap \dots \cap |D_{d+1}|.$$

Proof. Let

$$b : B \rightarrow X$$

denote the blowup of z , and let \mathcal{L} be the ample invertible sheaf given by $b^*\mathcal{O}_X(1)(-E)$, where E is the exceptional divisor. Then elements of $|\mathcal{L}^{\otimes m}|$ map to elements of $|\mathcal{O}_X(m)|$ which pass through z . Now choose m_0 such that $\mathcal{L}^{\otimes m}$ is very ample for $m \geq m_0$. Then by [13, 6.11(1)] we have that the intersection of $d + 1$ general elements of $|\mathcal{L}^{\otimes m}|$ is empty for $m \geq m_0$, and that general $D \in |\mathcal{L}^{\otimes m}|$ is irreducible. \square

Corollary 4.2.4. *Let X be an integral k -scheme and $\mathcal{O}_X(1)$ a very ample invertible sheaf on X . Given a regular closed point $z \in X$, we have that*

$$(4.2.4.1) \quad \{z\} = \bigcap |D| \subset |X|,$$

the intersection taken over all irreducible divisors D in $|\mathcal{O}_X(m)|$ for all m .

Proof. This follows from Lemma 4.2.3. \square

Proposition 4.2.5. *Suppose X and Y are definable schemes of dimension at least 2 with infinite constant fields, and assume that X is polarizable. Given an isomorphism $\varphi : \tau(X) \rightarrow \tau(Y)$, the associated homeomorphism $|X| \rightarrow |Y|$ extends to an isomorphism $X \rightarrow Y$.*

Proof. Let $D \in \text{Abpf}(X)$ be a divisor with $\mathcal{O}_X(D) = \mathcal{O}_X(1)$ and let $\mathcal{O}_Y(1)$ denote $\mathcal{O}_Y(\varphi(D))$. After possibly taking a power of our choice of polarization, we may assume that $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are very ample. Note that we are not asserting that we can detect very ampleness from $\tau(X)$ and $\tau(Y)$, just that we know that such a multiple must exist, so we are free to choose one.

By Summary 3.3.15 (iii), for each $m > 0$ the sets of definable lines are dense in the Grassmannians of $|\mathcal{O}_X(m)|$ and $|\mathcal{O}_Y(m)|$, contain the points of dense Zariski opens, and thus by Theorem 2.1.5, there is an isomorphism $\sigma_m : \kappa_X \rightarrow \kappa_Y$ and a σ -linear

isomorphism $\gamma_m : |\mathcal{O}_X(m)| \rightarrow |\mathcal{O}_Y(m)|$ that agrees with φ on a Zariski dense open subscheme $U \subset |\mathcal{O}_X(m)|$.

Consider the diagram of addition maps

$$\begin{array}{ccc} |\mathcal{O}_X(m)| & \longrightarrow & |\mathcal{O}_Y(m)| \\ +_X \uparrow & & +_Y \uparrow \\ |\mathcal{O}_X(1)|^{\times m} & \longrightarrow & |\mathcal{O}_Y(1)|^{\times m}. \end{array}$$

Since a general sum of divisors in $\mathcal{O}(1)$ lies in the sweep of the maximal open subset of the definable points by Lemma 4.2.2, we see that the associated diagram of schemes

$$(4.2.5.1) \quad \begin{array}{ccc} |\mathcal{O}_X(m)| & \xrightarrow{\gamma_m} & |\mathcal{O}_Y(m)| \\ +_X \uparrow & & +_Y \uparrow \\ |\mathcal{O}_X(1)|^{\times m} & \xrightarrow{\gamma_1^{\times m}} & |\mathcal{O}_Y(1)|^{\times m} \end{array}$$

commutes over the κ_X -points of $|\mathcal{O}_X(1)|^{\times m}$. Here the scheme $|\mathcal{O}_X(1)|^{\times m}$ is the m -fold fiber product of the projective space $|\mathcal{O}_X(1)|$ with itself over κ_X , and $|\mathcal{O}_Y(1)|^{\times m}$ is defined similarly over κ_Y .

Lemma 4.2.6. *The two isomorphisms of fields $\sigma_1, \sigma_m : \kappa_X \rightarrow \kappa_Y$ are equal.*

Proof. Let $U_1 \subset |\mathcal{O}_X(1)|$ (resp. $U_m \subset |\mathcal{O}_X(m)|$) be the sweep of the maximal open subset in the set of definable lines in $|\mathcal{O}_X(1)|$ (resp. $|\mathcal{O}_X(m)|$). Then $U_1^{\times m} \subset |\mathcal{O}_X(1)|^{\times m}$ is the κ_X -points of a nonempty Zariski open subset, and therefore

$$V := (+_X)^{-1}(U_m) \cap U_1^{\times m} \subset |\mathcal{O}_X(1)|^{\times m}$$

is the κ_X -points of a Zariski open subset of $|\mathcal{O}_X(1)|^{\times m}$. We can therefore find points

$$P, Q, R, P_2, \dots, P_m \in |\mathcal{O}_X(1)|$$

such that the three points of $|\mathcal{O}_X(1)|^{\times m}$ given by

$$(P, P_2, \dots, P_m), (Q, P_2, \dots, P_m), (R, P_2, \dots, P_m)$$

lie in V and $P, Q, R \in |\mathcal{O}_X(1)|$ are colinear. Since γ_1 and γ_m agree with the maps defined by φ on U_1 and U_m it follows that we have

$$\begin{aligned} (\gamma_m \circ +_X)(P, P_2, \dots, P_m) &= (+_Y \circ \gamma_1^{\times m})(P, P_2, \dots, P_m) \quad (\text{call this point } \overline{P} \in |\mathcal{O}_Y(m)|) \\ (\gamma_m \circ +_X)(Q, P_2, \dots, P_m) &= (+_Y \circ \gamma_1^{\times m})(Q, P_2, \dots, P_m) \quad (\text{call this point } \overline{Q} \in |\mathcal{O}_Y(m)|), \\ (\gamma_m \circ +_X)(R, P_2, \dots, P_m) &= (+_Y \circ \gamma_1^{\times m})(R, P_2, \dots, P_m) \quad (\text{call this point } \overline{R} \in |\mathcal{O}_Y(m)|), \end{aligned}$$

Let $\overline{L} \subset |\mathcal{O}_Y(m)|$ be the line through \overline{P} and \overline{Q} , and let $L \subset |\mathcal{O}_X(1)|$ be the line through P and Q . Then

$$+_X(L \times \{P_2\} \times \dots \times \{P_m\})$$

is the line in $|\mathcal{O}_X(m)|$ through the two points

$$+_X(P, P_2, \dots, P_m), \quad +_X(Q, P_2, \dots, P_m).$$

Since γ_m takes lines to lines it follows that

$$(\gamma_X \circ +_X)(L \times \{P_2\} \times \dots \times \{P_m\}) = \overline{L}.$$

Similarly since γ_1 takes lines to lines and agrees on U_1 with the map defined by φ , we find that

$$(+_Y \circ \gamma_1^{\times m})(L \times \{P_2\} \times \cdots \times \{P_m\}) = \bar{L}.$$

Since $\gamma_m \circ +_X$ and $+_Y \circ \gamma_1^{\times m}$ agree on a dense open subset of L , viewed as imbedded in $|\mathcal{O}_X(1)|^{\times m}$ via the identification

$$L \simeq L \times \{P_2\} \times \cdots \times \{P_m\}$$

we conclude that the two compositions

$$\kappa_X \xrightarrow{\alpha} L \subset |\mathcal{O}_X(1)|^{\times m} \xrightarrow{\gamma_m \circ +_X} \bar{L} \xrightarrow{\beta^{-1}} \kappa_Y$$

and

$$\kappa_X \xrightarrow{\alpha} L \subset |\mathcal{O}_X(1)|^{\times m} \xrightarrow{+_Y \circ \gamma_1^{\times m}} \bar{L} \xrightarrow{\beta^{-1}} \kappa_Y$$

agree on all but finitely many elements of κ_X , where $\alpha : \kappa_X \simeq L$ (resp. $\beta : \kappa_Y \simeq \bar{L}$) is the isomorphism obtained as in the proof of Theorem 2.1.5 using the three points P, Q, R (resp. $\bar{P}, \bar{Q}, \bar{R}$). Now the first of these maps is the map σ_m and the second is σ_1 . We conclude that $\sigma_1(a) = \sigma_m(a)$ for all but finitely many elements $a \in \kappa_X$, which implies that $\sigma_1 = \sigma_m$. \square

In the rest of the proof we write $\sigma : \kappa_X \rightarrow \kappa_Y$ for the isomorphism $\sigma_m = \sigma_1$.

Next observe that the diagram of schemes (4.2.5.1) commutes, since the two morphisms obtained by going around the different directions of the diagram are semi-linear with respect to the same field isomorphism and agree on dense set of points.

Consider the embeddings

$$\nu_X : X \hookrightarrow |\mathcal{O}_X(1)|^\vee$$

and

$$\nu_Y : Y \hookrightarrow |\mathcal{O}_Y(1)|^\vee$$

and let \bar{X} (resp. \bar{Y}) be the scheme-theoretic closure of $\nu_X(X)$ (resp. $\nu_Y(Y)$). Let S_X (resp. S_Y) be the symmetric algebra on $\Gamma(X, \mathcal{O}_X(1))$ (resp. $\Gamma(Y, \mathcal{O}_Y(1))$) so \bar{X} (resp. \bar{Y}) is given by a graded ideal $I_{\bar{X}} \subset S_X$ (resp. $I_{\bar{Y}} \subset S_Y$).

Choosing a lift

$$\tilde{\gamma}_1 : \Gamma(X, \mathcal{O}_X(1)) \rightarrow \Gamma(Y, \mathcal{O}_Y(1))$$

yields an induced σ -linear isomorphism of graded rings

$$\gamma^\natural : S_X \rightarrow S_Y$$

that is uniquely defined up to scalars. We claim that $\gamma^\natural(I_{\bar{X}}) = I_{\bar{Y}}$.

For this, consider the diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X(m)) & \xrightarrow{\tilde{\gamma}_m} & \Gamma(Y, \mathcal{O}_Y(m)) \\ p_X \uparrow & & \uparrow p_Y \\ S_X^m = \Gamma(X, \mathcal{O}_X(1))^{\otimes m} & \xrightarrow{\gamma_m^\natural} & \Gamma(Y, \mathcal{O}_Y(1))^{\otimes m} = S_Y^m \\ \uparrow & & \uparrow \\ \Gamma(X, \mathcal{O}_X(1))^{\times m} & \xrightarrow{\tilde{\gamma}^{\times m}} & \Gamma(Y, \mathcal{O}_Y(1))^{\times m} \end{array}$$

arising as follows. The vertical arrows are the natural multiplication maps, and the induced linear maps from the universal property of \otimes . The arrow $\tilde{\gamma}_m$ is a lift of γ_m . By the commutativity of diagram (4.2.5.1), we see that this diagram commutes (up to suitably scaling $\tilde{\gamma}_m$), which implies that $\gamma_m^\sharp(I_{\bar{X},m}) = I_{\bar{Y},m}$, as desired.

In summary, we have shown that if

$$A_{\bar{X}} \subset \bigoplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(m)) \quad (\text{resp. } A_{\bar{Y}} \subset \bigoplus_{m \geq 0} \Gamma(Y, \mathcal{O}_Y(m)))$$

denotes the subring generated by $\Gamma(X, \mathcal{O}_X(1))$ (resp. $\Gamma(Y, \mathcal{O}_Y(1))$), then we have an isomorphism of graded rings

$$\tilde{\gamma} : A_{\bar{X}} \rightarrow A_{\bar{Y}}$$

such that the isomorphism induced by $\tilde{\gamma}$ in degree m

$$|\mathcal{O}_{\bar{X}}(m)| \rightarrow |\mathcal{O}_{\bar{Y}}(m)|$$

fits into a commutative diagram

$$\begin{array}{ccc} |\mathcal{O}_{\bar{X}}(m)| & \longrightarrow & |\mathcal{O}_{\bar{Y}}(m)| \\ \downarrow & & \downarrow \\ |\mathcal{O}_X(m)| & \xrightarrow{\varphi} & |\mathcal{O}_Y(m)|, \end{array}$$

where the vertical maps are the restriction maps and the map labelled φ is the map given by the isomorphism of Torelli structures.

In other words, if we let

$$f : \bar{X} \rightarrow \bar{Y}$$

be the isomorphism given by $\tilde{\gamma}$, then the diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{f} & \bar{Y} \\ \wr \downarrow & & \downarrow \wr \\ \text{Proj}(A_{\bar{X}}) & \xrightarrow{\tilde{\gamma}} & \text{Proj}(A_{\bar{Y}}) \\ \downarrow & & \downarrow \\ |\mathcal{O}_X(1)|^\vee & \xrightarrow{\gamma_1^\vee} & |\mathcal{O}_Y(1)|^\vee \end{array}$$

commutes. The commutativity of the top square in this diagram implies that if $D \subset \bar{X}$ is an effective divisor in $|\mathcal{O}_{\bar{X}}(m)|$ then the image of the divisor $f(D) \subset \bar{Y}$ in $|\mathcal{O}_Y(m)|$ (i.e., the restriction $f(D)|_Y$) is the divisor $\gamma_m(D)$. In particular, if D is irreducible with generic point $\eta_D \in X$ then $f(\eta_D) = \varphi(\eta_D)$.

By Corollary 4.2.4 applied to \bar{X} , we conclude that f acts the same as φ on every regular closed point of X . Since $|X|$ is a Zariski topological space, it follows that φ and f have the same action on $|X^{\text{reg}}|$, the regular locus of X . This implies that there are open subschemes $U \subset X$ and $V \subset Y$ such that

- (1) $\text{codim}(U^c \subset X) \geq 2$,
- (2) $\text{codim}(V^c \subset Y) \geq 2$,
- (3) f induces an isomorphism $f|_U : U \rightarrow V$, and
- (4) $f|_U$ induces $\varphi|_U$ on topological spaces.

By Lemma 4.1.3, it follows that φ is algebraizable to a unique isomorphism $f : X \rightarrow Y$, showing that τ is fully faithful. \square

This completes the proof of Theorem 3.1.12.

4.3. The case of finite fields.

4.3.1. *Statements of results.* The main result of this section is the following.

Theorem 4.3.1.1. *Let X and Y be Cohen-Macaulay connected definable varieties over finite fields of dimension ≥ 3 . Any isomorphism $\varphi : \tau(X) \rightarrow \tau(Y)$ of Torelli structures is induced by a unique isomorphism of schemes $X \xrightarrow{\sim} Y$.*

The proof is based on the Bertini-Poonen theorem, generalized to complete intersections in [5] and reviewed in Section 4.3.2 below, and the probabilistic fundamental theorem of projective geometry, Theorem 2.3.1.

Remark 4.3.1.2. The Cohen-Macaulay assumptions in Theorem 4.3.1.1 are needed in order to apply known results on Bertini theorems over finite fields to deduce that, in a certain precise sense, a density 1 set of complete intersections in X and Y are reduced, the key point being that a generically smooth Cohen-Macaulay scheme is reduced. It might be possible to prove directly a Bertini theorem calculating the density of reduced hypersurfaces among all hypersurfaces, but for a non-Cohen-Macaulay scheme this could be strictly less than 1, which is not sufficient for our probabilistic fundamental theorem of projective geometry to apply.

4.3.2. *The Bertini-Poonen theorem.* In fact, we will not need the main results of [5,17], but only a certain key lemma. Poonen's argument, and its variant due to Bucur and Kedlaya, proceeds by treating points of small, medium, and large degrees separately. For our purposes, we will need only their results about points of large degree. We introduce some notation so that this result can be stated.

4.3.3. *The large degree estimate.* Let F be a finite field with q elements and let r and n be positive integers. Let X/F be a smooth finite type separated F -scheme of equidimension $m \geq r$ equipped with an embedding

$$X \hookrightarrow \mathbf{P}^n$$

defining an invertible sheaf $\mathcal{O}_X(1)$ on X .

Let S denote the polynomial ring in $n+1$ variables over F and let $S_d \subset S$ denote the degree d elements in this ring, so we have a ring homomorphism

$$S \rightarrow \Gamma_*(X, \mathcal{O}_X(1))$$

restricting to a map

$$S_d \rightarrow \Gamma(X, \mathcal{O}_X(d))$$

of vector spaces.

Fix functions

$$g_i : \mathbf{N} \rightarrow \mathbf{N}$$

for $i = 2, \dots, r$ such that there exists an integer $w > 0$ for which

$$d \leq g_i(d) \leq wd$$

for all $d \in \mathbb{N}$ and all i . For notational reasons it will be convenient to also write $g_1 : \mathbb{N} \rightarrow \mathbb{N}$ for the identity function.

4.3.3.1. Let \mathcal{S} denote the product $\prod_{j=1}^r S$ and let \mathcal{S}_d denote the subset

$$S_d \times S_{g_2(d)} \times \cdots \times S_{g_r(d)} \subset \mathcal{S}.$$

For a section $f \in S_d$ let $H_{X,f} \subset X$ be the closed subscheme defined by the image of f in $\Gamma(X, \mathcal{O}_X(d))$, and for $\underline{f} = (f_1, \dots, f_r) \in \mathcal{S}_d$ let

$$(4.3.3.1.1) \quad X_{\underline{f}} := \bigcap_{i=1}^r H_{X,f_i}$$

For an integer d let $W_d \subset \mathcal{S}_d$ denote the subset of vectors \underline{f} such that the intersection $X_{\underline{f}}$ is smooth of dimension $m - r$ at all closed points P in this intersection of degree $> d/(m + 1)$, and define

$$e_d := 1 - \frac{\#W_d}{\#\mathcal{S}_d}.$$

Lemma 4.3.3.2. *There exists a constant C , depending on n, r, m, w , and the degree of $\bar{X} \subset \mathbb{P}^n$, such that*

$$e_d \leq Cd^m q^{-\min\{d/(m+1), d/p\}}.$$

In particular,

$$\lim_{d \rightarrow \infty} e_d = 0.$$

Proof. This is [5, 2.7]. □

We recall some additional useful notation from [17], which we will use in stating consequences of Lemma 4.3.3.2. For a subset $\mathcal{P} \subset \mathcal{S}$ write

$$\mu(\mathcal{P}) := \lim_{d \rightarrow \infty} \frac{\#(\mathcal{S}_d \cap \mathcal{P})}{\#\mathcal{S}_d}$$

and

$$\bar{\mu}(\mathcal{P}) := \limsup_{d \rightarrow \infty} \frac{\#(\mathcal{S}_d \cap \mathcal{P})}{\#\mathcal{S}_d}$$

4.3.4. Variants.

4.3.4.1. As in Section 4.3.3, let \mathbb{F} be a finite field and let $X \subset \mathbb{P}^n$ be a quasi-projective scheme. Assume that X is reduced and Cohen-Macaulay, and of equidimension m and that $r < m$.

Let $H_d \subset \mathcal{S}_d$ be the subset of elements (f_1, \dots, f_r) such that for every subset $R \subset \{1, \dots, r\}$ the scheme-theoretic intersection

$$X_R := \bigcap_{i \in R} X_{f_i}$$

is reduced of dimension $m - \#R$. Let $H \subset \mathcal{S}$ denote the union of the H_d .

Theorem 4.3.4.2. *We have*

$$\mu(H) = 1.$$

Proof. For a given R , let $H_{R,d} \subset \mathcal{S}_d$ be the subset of those vectors for which the intersection X_R is reduced of dimension $m - \#R$, and let H_R denote the union of the $H_{R,d}$. Then

$$1 - \frac{\#H_d}{\#\mathcal{S}_d} \leq \sum_R \left(1 - \frac{\#H_{R,d}}{\#\mathcal{S}_d} \right).$$

It therefore suffices to show that $\mu(H_R) = 1$. Furthermore, this case reduces immediately to the case when $R = \{1, \dots, r\}$, which we assume henceforth.

Note that if the intersection X_R is generically smooth of the expected dimension, then it is reduced. Indeed this follows from the fact that a complete intersection in a Cohen-Macaulay scheme is Cohen-Macaulay and that a generically reduced Cohen-Macaulay scheme is reduced.

Let $\overline{X} \subset \mathbf{P}^n$ be the closure of X with the reduced scheme structure, and fix a finite stratification $\overline{X} = \{X_i\}_{i \in I}$ with each X_i a smooth locally closed subscheme of \overline{X} , and one of the strata X_0 equal to the smooth locus of X . If we further arrange that each $X_{i,R} \subset X_i$ has the expected dimension then it follows that the inclusion

$$X_R \cap X_0 \hookrightarrow X_R$$

is dense.

For an integer s let $E_{X_i,d}^{(s)} \subset \mathcal{S}_d$ denote the subset of those vectors (f_1, \dots, f_r) for which the intersections $X_{i,\underline{f}}$ is smooth of the expected dimension at all points P of degree $\geq s$. Let $E_d^{(s)}$ denote the intersection of the $E_{X_i,d}^{(s)}$.

Observe that since we assumed that $r < m$, the closed points of degree $\geq s$ are dense in any irreducible component of $X_{0,\underline{f}}$. In particular, we have $E_d^{(s)} \subset H_d$. Let $E^{(s)}$ denote the union of the $E_d^{(s)}$. Taking $s = \lfloor \frac{d}{m+1} + 1 \rfloor$ have

$$E_{X_i,d}^{(s)} = W_{X_i,d},$$

where $W_{X_i,d}$ is defined as in Paragraph 4.3.3.1 applied to X_i .

By this and Lemma 4.3.3.2 we have that

$$\lim_{d \rightarrow \infty} \frac{\sum_i \#(\mathcal{S}_d \setminus E_{X_i,d})}{\#\mathcal{S}_d} = 0.$$

We conclude that

$$\lim_{d \rightarrow \infty} \frac{\#E_d^{(s)}}{\#\mathcal{S}_d} = 1,$$

and it follows that

$$\mu(H_R) = 1,$$

as desired. □

4.3.5. Preparatory lemmas. We continue with the setup of Section 4.3.3.

Lemma 4.3.5.1. *Let k be a field and let \overline{D}/k be a geometrically irreducible proper k -scheme, let $D \subset \overline{D}$ be a dense open subscheme with D geometrically reduced and $\text{codim}(\overline{D} - D, \overline{D}) \geq 2$. Then $H^0(D, \mathcal{O}_D) = k$.*

Proof. We may without loss of generality assume that \overline{D} is reduced, and furthermore, by replacing \overline{D} by its normalization that \overline{D} is normal. Since \overline{D} is geometrically reduced and irreducible it follows that $H^0(\overline{D}, \mathcal{O}_{\overline{D}}) = H^0(D, \mathcal{O}_D) = k$. \square

Lemma 4.3.5.2. *Let X be a definable Cohen-Macaulay variety over a perfect field k and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf with associated linear system P . Let $j : X \hookrightarrow \overline{X}$ be the compactification of X provided by the given projective embedding, so X is schematically dense in \overline{X} . Fix a finite stratification $\{\overline{X}_i\}_{i \in I}$ of \overline{X} with each \overline{X}_i smooth, equidimensional, and $\overline{X}_i \hookrightarrow \overline{X}$ locally closed. Let $F, G \in H^0(X, \mathcal{O}_X(1)) = H^0(\overline{X}, \mathcal{O}_{\overline{X}}(1))$ be two linearly independent sections. Let \overline{D}_F (resp. \overline{D}_G) be the zero locus in \overline{X} of F (resp. G) and set $D_F := \overline{D}_F \cap X$ (resp. $D_G := \overline{D}_G \cap X$), and assume they satisfy the following:*

- (i) \overline{D}_F is geometrically irreducible.
- (ii) *The intersection $\overline{D}_F \cap \overline{X}_i$ has dimension $\dim(\overline{X}_i) - 1$ for all i , and the intersection $\overline{D}_F \cap \overline{D}_G \cap \overline{X}_i$ has dimension $\dim(\overline{X}_i) - 2$ for all i (here we make the convention that the empty scheme has dimension -1 as well as -2).*
- (iii) D_F and the intersection $D_F \cap D_G$ are generically smooth.

Then F and G span a definable line.

Proof. Assumption (ii) implies that $D_F \subset \overline{D}_F$ is dense. Furthermore, D_F is Cohen-Macaulay (see for example [?, Tag 02JN]) and the generic smoothness of D_F , assumed in (iii), then implies that D_F is geometrically reduced. Similarly, $D_F \cap D_G$ is geometrically reduced, and $D_F \cap D_G \subset \overline{D}_F \cap \overline{D}_G$ is dense. We need to show that the kernel of the restriction map

$$H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(D_F \cap D_G, \mathcal{O}_{D_F \cap D_G}(1))$$

is the span of F and G .

To this end, let W denote $(\overline{D}_F \cap \overline{D}_G) \setminus (D_F \cap D_G)$. By assumption (ii), W has codimension at least 2 in \overline{D}_F , and we have a closed immersion

$$D_F \cap D_G \hookrightarrow \overline{D}_F \setminus W.$$

From this we therefore get an exact sequence

$$0 \longrightarrow H^0(\overline{D}_{F,\text{red}} \setminus W, \mathcal{O}_{\overline{D}_{F,\text{red}}} \otimes \mathcal{O}_{\overline{D}_{F,\text{red}} \setminus W}) \xrightarrow{G} H^0(\overline{D}_{F,\text{red}} \setminus W, \mathcal{O}_{\overline{D}_{F,\text{red}}} \otimes \mathcal{O}_{\overline{D}_{F,\text{red}} \setminus W}(1)) \longrightarrow H^0(D_F \cap D_G, \mathcal{O}_{D_F \cap D_G}(1)).$$

From the commutative diagram

$$\begin{array}{ccc} H^0(\overline{X}, \mathcal{O}_{\overline{X}}(1)) & \xrightarrow{\cong} & H^0(X, \mathcal{O}_X(1)) \\ \downarrow & & \downarrow \\ H^0(\overline{D}_{F,\text{red}} \setminus W, \mathcal{O}_{\overline{D}_{F,\text{red}} \setminus W}(1)) & \longrightarrow & H^0(D_F, \mathcal{O}_{D_F}(1)) \end{array}$$

we see that the kernel of the map

$$H^0(\overline{X}, \mathcal{O}_{\overline{X}}(1)) \rightarrow H^0(\overline{D}_{F,\text{red}} \setminus W, \mathcal{O}_{\overline{D}_{F,\text{red}} \setminus W}(1))$$

is 1-dimensional generated by F . From this and the argument used in Lemma 3.3.11 we see that to prove the proposition it suffices to show that the dimension of $H^0(\overline{D}_{F,\text{red}} \setminus W, \mathcal{O}_{\overline{D}_{F,\text{red}} \setminus W}(1))$ is 1. This follows from Lemma 4.3.5.1. \square

Lemma 4.3.5.3. *Let X/k be a Cohen-Macaulay quasi-projective definable variety of dimension at least 3, and let $\mathcal{O}_X(1)$ be a very ample line bundle on X . Let $X \hookrightarrow \mathbf{P}$ be the embedding into projective space provided by $\mathcal{O}_X(1)$ and let $\overline{X} \subset \mathbf{P}$ be the closure of X , with the reduced subscheme structure. Let \mathcal{H}_n be the set of lines in $\mathbf{P}H^0(\overline{X}, \mathcal{O}_{\overline{X}}(n))$ and let $\mathcal{H}_n^{\text{def}} \subset \mathcal{H}_n$ be the subset of lines which are definable as lines in $\mathbf{P}H^0(X, \mathcal{O}_X(n))$. Then*

$$\lim_{n \rightarrow \infty} \mu_{\mathcal{H}_n}(\mathcal{H}_n^{\text{def}}) = 1.$$

Proof. Fix a finite stratification $\{X_i\}_{i \in I}$ of \overline{X} into locally closed smooth subschemes.

Let \mathcal{P}_n denote the set of pairs of linearly independent elements

$$f_1, f_2 \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(n)),$$

and let $\mathcal{P}'_n \subset \mathcal{P}_n$ denote the subset of pairs (f_1, f_2) with associated divisors $\overline{D}_s := V(f_s) \cap \overline{X}$ have the following properties:

- (i) \overline{D}_s is geometrically irreducible for $s = 1, 2$.
- (ii) $\overline{D}_s \cap \overline{X}_i$ and the double intersections $\overline{D}_1 \cap \overline{D}_2 \cap X_i$ have the expected dimension.
- (iii) $\overline{D}_1 \cap X$, $\overline{D}_2 \cap X$, and $\overline{D}_1 \cap \overline{D}_2 \cap X$ are generically smooth.

There is a map

$$Sp : \mathcal{P}_n \rightarrow \mathcal{H}_n$$

sending a pair (f_1, f_2) to the line spanned by f_1 and f_2 . By Lemma 4.3.5.2 the image of \mathcal{P}'_n is contained in $\mathcal{H}_n^{\text{def}}$ and therefore we have

$$\mu_{\mathcal{P}_n}(\mathcal{P}'_n) \leq \mu_{\mathcal{H}_n}(\mathcal{H}_n^{\text{def}}),$$

and it suffices to show that

$$\lim_{n \rightarrow \infty} \mu_{\mathcal{P}_n}(\mathcal{P}'_n) = 1.$$

This follows from Theorem 4.3.4.2 and [8, Theorem 1.1]. □

4.3.5.4. For integers n_1, n_2 with $n_1 \leq n_2 \leq 2n_1$ consider the subset

$$\mathcal{T}_{n_1, n_2} \subset S_{n_1} \oplus S_{n_2} \oplus S_{n_1 + n_2}$$

whose elements are triples (f_1, f_2, f_3) for which the elements $f_1 f_2$ and f_3 span a definable line in $\mathbf{P}\Gamma(X, \mathcal{O}_X(n_1 + n_2))$.

Lemma 4.3.5.5. *For any function $g : \mathbf{N} \rightarrow \mathbf{N}$ such that $n \leq g(n) \leq 2n$ for all n , we have*

$$\lim_{n \rightarrow \infty} \frac{\#\mathcal{T}_{n, g(n)}}{\#(S_n \oplus S_{g(n)} \oplus S_{n+g(n)})} = 1.$$

Proof. Fix a stratification $\{X_i\}_{i \in I}$ of \overline{X} into smooth subschemes.

By Lemma 4.3.5.2 the set $\mathcal{T}_{n, g(n)}$ contains the set $\mathcal{T}'_{n, g(n)}$ of triples $(f_1, f_2, f_3) \in S_n \oplus S_{g(n)} \oplus S_{n+g(n)}$ satisfying the condition that the zero locus of each f_i in \overline{X} is irreducible, and for all $R \subset \{1, 2, 3\}$ the intersection X_R is generically smooth and the intersection $\overline{X}_R \cap X_i$ have the expected dimension for all i . The result then follows from Theorem 4.3.4.2 and [8, Theorem 1.1] □

4.3.6. *Proof of Theorem 4.3.1.1.* By the same argument as in Section 4.1, which did not require any assumption on the ground field, it suffices to prove Theorem 4.3.1.1 in the case when X and Y are quasi-projective.

4.3.6.1. Fix $\epsilon > 0$. In the course of the proof we will make various assumptions on ϵ being sufficiently small. As there are only finitely many steps, this is a harmless practice.

Fix an ample invertible sheaf $\mathcal{O}_X(1)$ on X represented by an effective divisor D . By Proposition 3.2.7 the property of being ample depends only on the Torelli structure, and therefore $\varphi(D)$ defines an ample invertible sheaf on Y , which we denote by $\mathcal{O}_Y(1)$. After replacing $\mathcal{O}_X(1)$ by $\mathcal{O}_X(n)$ for sufficiently large n we may assume that $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are very ample.

Furthermore, by choosing n sufficiently large we may assume by Lemma 4.3.5.3 that there exist definable lines in $\mathbf{P}\Gamma(X, \mathcal{O}_X(1))$ and $\mathbf{P}\Gamma(Y, \mathcal{O}_Y(1))$. Since the number of elements in a definable line is $q + 1$, we see that the finite fields κ_X and κ_Y are isomorphic to the same finite field \mathbf{F} , and in particular have the same number of elements which we will denote by q .

4.3.6.2. Let $\overline{X} \subset \mathbf{P}\Gamma(X, \mathcal{O}_X(1))$ (resp. $\overline{Y} \subset \mathbf{P}\Gamma(Y, \mathcal{O}_Y(1))$) be the scheme-theoretic closure of X (resp. Y). Define graded rings

$$A_{\overline{X}} := \bigoplus_{n \geq 0} \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(n)), \quad A_{\overline{Y}} := \bigoplus_{n \geq 0} \Gamma(\overline{Y}, \mathcal{O}_{\overline{Y}}(n)),$$

$$A_X := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)), \quad A_Y := \bigoplus_{n \geq 0} \Gamma(Y, \mathcal{O}_Y(n)),$$

so $A_{\overline{X}} \subset A_X$ and $A_{\overline{Y}} \subset A_Y$. For $m > 0$ and any of these graded rings A write $A(m)$ for the subring $A(m) \subset A$ given by

$$A(m) := \bigoplus_{n \geq 0} A^{nm}.$$

Write $|A_{\overline{X}}^n| \subset |nD|$ for $\mathbf{P}\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(n)) \subset \mathbf{P}\Gamma(X, \mathcal{O}_X(n))$, and similarly for $|A_{\overline{Y}}^n|$. By Lemma 4.3.5.3, for n sufficiently large the proportion of definable lines in the linear system $|A_{\overline{X}}^n|$ is greater than or equal to $1 - \epsilon$. Choosing ϵ sufficiently small, and thereafter n sufficiently large so that Theorem 2.3.1 applies to the map

$$|A_{\overline{X}}^n| \hookrightarrow \mathbf{P}\Gamma(X, \mathcal{O}_X(n)) \rightarrow \mathbf{P}\Gamma(Y, \mathcal{O}_Y(n))$$

we therefore find an integer n_0 such that for each $n \geq n_0$ we get an isomorphism of fields

$$\sigma_n : \kappa_X \rightarrow \kappa_Y$$

and a σ_n -linear

$$\gamma_n : A_{\overline{X}}^n \rightarrow \Gamma(Y, \mathcal{O}_Y(n))$$

such that the induced morphism of projective spaces

$$f'_n : \mathbf{P}\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(n)) \rightarrow \mathbf{P}\Gamma(Y, \mathcal{O}_Y(n))$$

agrees with the map

$$f_n : |A_{\overline{X}}^n| \rightarrow |\mathcal{O}_Y(n)|$$

defined by φ on a proportion of points

$$(4.3.6.2.1) \quad 1 - 2\epsilon - 2A(q, N, \epsilon) - 2 \frac{q^2(q+1)^2}{q^{N+1} - q},$$

where N is the projective dimension of $|A_X^n|$.

For notational convenience let $\tilde{\epsilon}$ denote 9ϵ . Then for n sufficiently large the expression (4.3.6.2.1) is greater than or equal to $1 - \tilde{\epsilon}$. After possibly replacing n_0 by a bigger number we may assume that (4.3.6.2.1) is at least $1 - \tilde{\epsilon}$ for all $n \geq n_0$.

After possibly replacing n_0 by a bigger number and applying Lemma 4.3.5.5, we may assume the following.

Assumption 4.3.6.3. For all $n \geq n_0$, we have that

$$\frac{\#\mathcal{T}_{n,g(n)}}{\#(S_n \oplus S_{g(n)} \oplus S_{n+g(n)})} = 1 - \tilde{\epsilon}$$

for $g(n) = n$ and $g(n) = n + 1$.

4.3.6.4. Next we prove that the f'_n are close to multiplicative.

Claim 4.3.6.5. Given Assumption 4.3.6.3, for any $n_1 \geq n_0$ and for $n_2 = n_1$ or $n_2 = n_1 + 1$ we have

$$f'_{n_1}(s_1)f'_{n_2}(s_2) = f'_{n_1+n_2}(s_1s_2)$$

for a proportion $1 - 5\tilde{\epsilon}$ of pairs

$$(s_1, s_2) \in A_X^{n_1} \times A_X^{n_2}.$$

Proof. Let Ξ denote the set of pairs (s_1, s_2) such that the proportion of lines through s_1s_2 that are definable is at most $1/4$. The hypothesis of the Claim and Lemma 4.3.5.5 imply that

$$\#\Xi \cdot \frac{1}{4} \#S_{n_1+n_2} \leq \tilde{\epsilon} \cdot \#S_{n_1} \cdot \#S_{n_2} \cdot \#S_{n_1+n_2}.$$

It follows that for a proportion of $1 - 4\tilde{\epsilon}$ of pairs (s_1, s_2) , the line in $\Gamma(X, \mathcal{O}_X(n_1 + n_2))$ spanned by s_1s_2 and at least three quarters of the elements in $A_X^{n_1+n_2}$ is definable. Thus, for at least half of the pairs (s_3, s'_3) of elements in $A_X^{n_1+n_2}$, the lines $Sp(s_1s_2, s_3)$ and $Sp(s_1s_2, s'_3)$ are definable.

Consider the set Λ of lines ℓ through s_1s_2 such that $f_{n_1+n_2}(t) \neq f'_{n_1+n_2}(t)$ for all t in ℓ except possibly for one point; let $\tilde{\Lambda}$ denote the set of all lines through s_1s_2 . A straightforward counting argument shows that $\#\Lambda/\#\tilde{\Lambda} < 4\tilde{\epsilon}$. So assuming $4\tilde{\epsilon} < 1/4$, for a proportion $1 - 4\tilde{\epsilon}$ of pairs (s_1, s_2) we can find a pair (s_3, s'_3) such that the lines

$$Sp(s_1s_2, s_3), \quad Sp(s_1s_2, s'_3)$$

are definable and each contain at least two points for which $f_n = f'_n$. Since f_n preserves definable lines, and f'_n takes lines to lines, we conclude that for such (s_3, s'_3) we have

$$f'_n(Sp(s_1s_2, s_3)) = f_n(Sp(s_1s_2, s_3)), \quad f'_n(Sp(s_1s_2, s'_3)) = f_n(Sp(s_1s_2, s'_3)).$$

Since the intersection of these two lines is s_1s_2 we conclude that

$$f'_n(s_1s_2) = f_n(s_1s_2) = f_n(s_1)f_n(s_2).$$

Since $f_n = f'_n$ on a proportion of $1 - \tilde{\epsilon}$ points we conclude that $f'_{n_1}(s_1)f'_{n_2}(s_2) = f'_{n_1+n_2}(s_1s_2)$ for a proportion of $1 - 5\tilde{\epsilon}$ pairs. \square

4.3.6.6. Next we show that the γ_n are close to multiplicative.

Claim 4.3.6.7. *Given Assumption 4.3.6.3, for any n_1 and $n_2 = n_1$ or $n_2 = n_1 + 1$, there exists a constant c_{n_1, n_2} such that*

$$(4.3.6.7.1) \quad \gamma_{n_1}(s_1)\gamma_{n_2}(s_2) = c_{n_1, n_2}\gamma_{n_1+n_2}(s_1s_2)$$

for all pairs (s_1, s_2) .

Proof. Let V_n denote $A_{\overline{X}}^n$, viewed as a vector space over the prime field \mathbb{F}_p . We then have two symmetric bilinear forms

$$b_X, b_Y : V_{n_1} \times V_{n_2} \rightarrow \Gamma(Y, \mathcal{O}_Y(n_1 + n_2))$$

given by

$$b_X(s_1, s_2) := \gamma_{n_1}(s_1)\gamma_{n_2}(s_2), \quad b_Y(s_1, s_2) := \gamma_{n_1+n_2}(s_1s_2).$$

These forms have the property that they agree up to scalar for a proportion of $1 - 5\tilde{\epsilon}$ of pairs (s_1, s_2) .

Given s_1 , let Y_{s_1} be the set of s_2 such that $b_X(s_1, s_2)$ is a scalar multiple of $b_Y(s_1, s_2)$. Let $p(s_1) = \#Y_{s_1}/\#V_{n_2}$. It follows from the above remarks that we have

$$\#\{s_1 | p(s_1) < 2\sqrt{\tilde{\epsilon}}\} \cdot 2\sqrt{\tilde{\epsilon}} < 5\tilde{\epsilon} \cdot \#V_{n_1}.$$

Thus, for a proportion of $1 - (5/2)\sqrt{\tilde{\epsilon}}$ of elements s_1 the two forms $b_X(s_1, s_2)$ and $b_Y(s_1, s_2)$ agree up to scalar for a proportion of $1 - 2\sqrt{\tilde{\epsilon}}$ of elements s_2 .

Fix s_1 for which $p(s_1) \geq 2\sqrt{\tilde{\epsilon}}$. Each of the maps

$$b_X(s_1, -), b_Y(s_1, -) : V_{n_2} \rightarrow \Gamma(Y, \mathcal{O}_Y(n_1 + n_2))$$

are injective, which implies that

$$\text{rank}(b_X(s_1, -) - \alpha b_Y(s_1, -)) + \text{rank}(b_X(s_1, -) - \alpha' b_Y(s_1, -)) \geq \dim(V_{n_2})$$

for any distinct elements α, α' . It follows that there is at most one $\alpha \neq 0$ for which the rank of $b_X(s_1, -) - \alpha b_Y(s_1, -)$ is less than or equal to $\dim(V_{n_2})/2$.

Suppose that in fact we have

$$\text{rank}(b_X(s_1, -) - \alpha b_Y(s_1, -)) \geq \dim(V_{n_2})/2$$

for all α . Then the proportion of s_2 for which $b_X(s_1, s_2)$ is a scalar multiple of $b_Y(s_1, s_2)$ is at most

$$\frac{q-1}{q^{\dim(V_{n_2})/2}},$$

and we obtain the inequality

$$\frac{q-1}{q^{\dim(V_{n_2})/2}} \geq 1 - 2\sqrt{\tilde{\epsilon}}.$$

For n chosen sufficiently large relative to ϵ this is a contradiction. We conclude that there exists exactly one scalar α_0 such that

$$\text{rank}(b_X(s_1, -) - \alpha_0 b_Y(s_2, -)) < \dim(V_{n_2})/2.$$

Now in this case we find that the proportion of s_2 for which $b_X(s_1, s_2)$ is a scalar multiple of $b_Y(s_1, s_2)$ is at most

$$\frac{(q-2)}{q^{\dim(V_{n_2})/2}} + \frac{1}{p^{r_0}},$$

where r_0 is the rank of $b_X(s_1, -) - \alpha_0 b_Y(s_1, -)$. For ϵ suitably small we see that this implies that in fact $r_0 = 0$ and $b_X(s_1, s_2) = \alpha_0 b_Y(s_1, s_2)$ for all s_2 .

Note that this argument is symmetric in s_1 and s_2 . That is, for a fixed s_2 subject to the condition that $b_X(s_1, s_2)$ is a scalar multiple of $b_Y(s_1, s_2)$ is at least $1 - 2\sqrt{\epsilon}$ we find that there exists a constant β such that

$$b_X(s_1, s_2) = \beta b_Y(s_1, s_2)$$

for all s_1 . From this it follows that in fact the constant α_0 in the previous paragraph is independent of the choice of s_1 . Furthermore, using the bilinearity we find that there exists a constant c_{n_1, n_2} such that

$$b_X(s_1, s_2) = c_{n_1, n_2} b_Y(s_1, s_2)$$

for all pairs (s_1, s_2) . In other words, we have the equality (4.3.6.7.1) □

Claim 4.3.6.8. *Given Assumption 4.3.6.3, for every $n \geq n_0$ and integer $m \geq 1$ there exists a constant c_m such that for all sections*

$$s_1, \dots, s_m \in A_{\overline{X}}^n$$

we have

$$\gamma_{nm}(s_1 \cdots s_m) = c_m \gamma_n(s_1) \cdots \gamma_n(s_m).$$

Proof. This we show by induction, the case $m = 1$ being vacuous. For the inductive step write $m = a + b$ for positive integers a and b with $a = b = m/2$ if m is even, and $a = (m - 1)/2$ and $b = (m + 1)/2$ if m is odd. Then by the above discussion there exists a constant $c_{a,b}$ such that

$$\gamma_{nm}(s_1 \cdots s_m) = c_{a,b} \gamma_{na} \left(\prod_{i=1}^a s_i \right) \gamma_{bn} \left(\prod_{j=1}^b s_{a+j} \right).$$

By our inductive hypothesis this equals

$$c_{a,b} c_a c_b \gamma_n(s_1) \cdots \gamma_n(s_m),$$

so we can take $c_m = c_{a,b} c_a c_b$. □

4.3.6.9. In particular, after possibly choosing n_0 even bigger so that $A_{\overline{X}}(n_0)$ is generated by $A_{\overline{X}}^{n_0}$ we get an injective ring homomorphism

$$\rho_{\overline{X}, n_0} : A_{\overline{X}}(n_0) \rightarrow A_Y(n_0)$$

given in degree mn_0 by γ_{mn_0}/c_m .

4.3.6.10. The map $\rho_{\overline{X}, n_0}$ defines a rational map

$$\lambda : Y \dashrightarrow \overline{X}.$$

Let $Y^\circ \subset Y$ be the maximal open subset over which λ is defined and the map $\rho_{\overline{X}, n_0}$ induces an isomorphism $\lambda^* \mathcal{O}_{\overline{X}}(n_0) \simeq \mathcal{O}_Y(n_0)$. We claim that the two maps of topological spaces

$$(4.3.6.10.1) \quad |\lambda|, \varphi^{-1} : |Y^\circ| \rightarrow |\overline{X}|$$

agree, where we write φ^{-1} also for the composition

$$|Y| \xrightarrow{\varphi^{-1}} |X| \hookrightarrow |\overline{X}|$$

To prove this it suffices to show that these two maps agree on all closed points. Suppose to the contrary that we have a closed point $y \in Y^\circ$ such that $\lambda(y) \neq \varphi^{-1}(y)$. Consider the subset

$$T_m \subset A_{\overline{X}}^{n_0 m}$$

of sections $g \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(n_0 m))$ whose zero locus contains both $\lambda(y)$ and $\varphi^{-1}(y)$. Now any section g whose zero locus contains $\lambda(y)$ and for which $f_{n_0 m}(g) = f'_{n_0 m}(g)$ lies in T_m by definition of λ and f_n . It follows that

$$(4.3.6.10.2) \quad \frac{\#T_m}{\#A_{\overline{X}}^{n_0 m}} \geq \frac{1}{q^{\deg(\lambda(y))}} - q^{\deg(\lambda(y))} \tilde{\epsilon}.$$

On the other hand, for m sufficiently big we have

$$(4.3.6.10.3) \quad \frac{\#T_m}{\#A_{\overline{X}}^{n_0 m}} = \frac{1}{q^{\deg(\lambda(y)) + \deg(\varphi^{-1}(y))}}.$$

Now observe that if we replace our choice of n_0 by a multiple, the open subset $Y^\circ \subset Y$ and λ remain the same, but we can decrease the size of $\tilde{\epsilon}$ by making such a choice of n_0 . Since the right side of Eq. (4.3.6.10.2) is larger than the right side of Eq. (4.3.6.10.3) for $\tilde{\epsilon}$ sufficiently small this gives a contradiction. We conclude that the two maps Eq. (4.3.6.10.1) agree.

Lemma 4.3.6.11. *Let k be a field, let S be a normal quasi-projective k -scheme, and let T/k be a proper k -scheme. Let $f : |S| \rightarrow |T|$ be a continuous map of topological spaces which is a homeomorphism onto an open subset of $|T|$. Assume that there exists a dense open subset $U \subset S$ and a morphism of schemes $\tilde{f}_U : U \rightarrow T$ whose underlying morphism of topological spaces $|\tilde{f}_U| : |U| \rightarrow |T|$ agrees with the restriction of f . Then there exists a unique morphism of schemes $f : S \rightarrow T$ whose underlying morphism of topological spaces is f and which restricts to \tilde{f}_U on U .*

Proof. Since S is normal and T is proper there exists an open subset $S^\circ \subset S$ containing U and with complement of codimension ≥ 2 such that \tilde{f}_U extends to a morphism of schemes

$$\tilde{f}_{S^\circ} : S^\circ \rightarrow T.$$

We claim that \tilde{f}_{S° induces $f|_{|S^\circ|}$ on underlying topological spaces.

If $s \in S^\circ$ is a closed point, then by Bertini's theorem (or in the finite field case Poonen-Bertini) there exist effective irreducible divisors $D_1, \dots, D_r \subset S^\circ$, with $D_i \cap U$ nonempty for all i , such that

$$\{s\} = D_1 \cap \dots \cap D_r.$$

Since f is a homeomorphism onto an open subset of T we can further arrange that

$$\{f(s)\} = \overline{f(D_1)} \cap \dots \cap \overline{f(D_r)},$$

where $\overline{f(D_i)}$ is the closure of $f(D_i)$ in $|T|$. Since

$$\tilde{f}_{S^\circ}(s) \subset \overline{f(D_1)} \cap \dots \cap \overline{f(D_r)}.$$

We conclude that $\tilde{f}_{S^\circ}(s) = f(s)$. This shows that \tilde{f}_{S° agrees with f on all closed points and therefore also on all points.

We are therefore reduced to the case when the complement of U in Y has codimension ≥ 2 . In this case, the morphism \tilde{f}_U extends to a map $\tilde{f} : Y \rightarrow T$ by the same argument as in the proof of Lemma 4.1.3, and repeating the previous argument we see that \tilde{f} induces f on topological spaces. \square

4.3.6.12. By Lemma 4.3.6.11 we therefore get a morphism of schemes

$$u : Y \rightarrow X$$

whose underlying morphism of topological spaces is φ^{-1} .

For n sufficiently big, the line bundle $\mathcal{O}_Y(n)$ can be represented by an effective divisor $D \subset Y$ all of whose irreducible components occur with multiplicity one and have nonempty intersection with Y° . The divisor $\varphi(D)$ then represents the line bundle $\mathcal{O}_X(n)$, and we have a nonzero map

$$u^*\mathcal{O}_X(-n) = u^*\mathcal{O}_X(-\varphi^{-1}(D)) \rightarrow \mathcal{O}_Y(-D) = \mathcal{O}_Y(-n).$$

Since φ induces an isomorphism on class groups we conclude that this map is an isomorphism, so u extends to a map of polarized schemes

$$u : (Y, \mathcal{O}_Y(n)) \rightarrow (X, \mathcal{O}_X(n)),$$

for all n sufficiently big.

Since the cardinalities of the linear systems $\Gamma(X, \mathcal{O}_X(n))$ and $\Gamma(Y, \mathcal{O}_Y(n))$ are the same for all n , we conclude that u induces an isomorphism of graded rings

$$A_X(n) \rightarrow A_Y(n)$$

for all n sufficiently big. This implies that u is an open immersion. Indeed if \overline{X} (resp. \overline{Y}) is the closure of X (resp. Y) in the projective imbedding defined by $\Gamma(X, \mathcal{O}_X(n))$ (resp. $\Gamma(Y, \mathcal{O}_Y(n))$) then we see that u induces an isomorphism between the homogeneous coordinate rings of \overline{X} and \overline{Y} , and therefore u is an open immersion inducing an isomorphism of topological spaces, whence an isomorphism.

This completes the proof of Theorem 4.3.1.1. \square

4.4. Counterexamples to weaker statements. One might wonder if the congruence relation on $\text{Eff}(X)$ is really necessary, or if the Zariski topological space itself might suffice to capture X . In one direction, we have the following.

Lemma 4.4.1. *Given two primes p and q and two smooth projective surfaces X over $\overline{\mathbb{F}}_p$ and Y over $\overline{\mathbb{F}}_q$, each of Picard number 1, any homeomorphism between a curve in X and a curve in Y extends to a homeomorphism $|X| \rightarrow |Y|$.*

Proof. The proof of is essentially contained in the proof of the final Proposition of [23], once one notes that the topological space of such a surface satisfies the axioms laid out in [23, Corollary 1] (even though they are stated there only for \mathbb{P}^2). \square

Remark 4.4.2. In particular, there are examples of surfaces of general type that are homeomorphic to the projective plane. Even more bizarrely, one can show that $\mathbb{P}_{\overline{\mathbb{F}}_p}^2$ is homeomorphic to $\mathbb{P}_{\overline{\mathbb{F}}_q}^2$ for any primes p and q . In other words, the linear equivalence

relation in the divisorial structure is necessary. The proofs of [23] are heavily reliant on working over the algebraic closure of a finite field.

4.5. Counterexamples in dimension 1. In this section we provide counterexamples to Theorem 3.1.12 for schemes of dimension 1 over arbitrary fields. This shows that the assumption of algebraically closed base fields in [25] is necessary and not simply an artifact of the use of model theory.

Proposition 4.5.1. *Fix a field K . For an integer $g > 1$ let $C_g \rightarrow \text{Spec } K_g$ be the generic curve of genus g (so that K_g is the function field of \mathcal{M}_g – the moduli stack over K classifying genus g curves). Then for any two integers $g, h > 1$ we have that $\tau(C_g)$ is (non-canonically) isomorphic to $\tau(C_h)$. In particular, Theorem 3.1.12 is false for curves over non-algebraically closed fields.*

Proof. By the Franchetta Conjecture (e.g., [18, Theorem 5.1]), we know that $\text{Pic}(C_g) = \mathbf{Z} \cdot [K_{C_g}]$ and similarly for h . The package $\tau(C_g)$ is equivalent to assigning to each closed point $c \in C_g$ the corresponding element $d(c) \in \mathbf{N}$. To produce an isomorphism between $\tau(C_g)$ and $\tau(C_h)$ it suffices to show that for each $n \in \mathbf{N}$, there is a bijection

$$S_g(n) := \{c \in C_g \mid d(c) = n\} \rightarrow \{c' \in C_h \mid d(c') = n\} =: S_h(n).$$

For each n , consider the linear system $|nK_{C_g}|$. By Riemann–Roch, this has projective dimension $(2n - 1)g - 2n$ for $n > 1$ and dimension $g - 1$ for $n = 1$. For each pair of positive integers a and b , the natural map

$$|aK_{C_g}| \times |bK_{C_g}| \rightarrow |(a + b)K_{C_g}|$$

has proper closed image (by a simple dimension count). We conclude that for each $a \in \mathbf{N}$, there is a Zariski open $U_a \subset |aK_{C_g}|$ whose points correspond precisely to $S_g(a)$. This shows that $S_g(a)$ has the same cardinality as K_g .

Choosing a bijection $K_g \rightarrow K_h$ thus establishes bijections $S_g(n) \rightarrow S_h(n)$ for all $n \in \mathbf{N}$, giving the desired result. \square

5. RESULTS OVER UNCOUNTABLE FIELDS OF CHARACTERISTIC 0

5.1. Linear equivalence over uncountable fields of characteristic 0. In this section, we show that over uncountable algebraically closed fields of characteristic 0, one can recover linear equivalence entirely topologically. This gives the following results.

Theorem 5.1.1. *If X is a normal, proper variety of dimension at least 2 over an uncountable algebraically closed field k of characteristic 0, then linear equivalence of divisors is determined by $|X|$.*

Theorem 5.1.2. *If X is a normal proper variety of dimension at least 2 over an uncountable algebraically closed field k of characteristic 0, then X is uniquely determined as a scheme by its underlying Zariski topological space $|X|$. It is also uniquely determined as a scheme by its associated étale topos.*

It is interesting to compare this to work of Voevodsky on an anabelian-type conjecture of Grothendieck [22]. While Grothendieck’s conjecture and Voevodsky’s theorem

require a finitely generated base field, they also give more, namely full functoriality, not merely isomorphism.

It is also interesting to compare this to [11]. While it is not entirely clear how the present work relates to the theory of Zariski geometries, we believe that it should be relatively straightforward to derive [11, Proposition 1.1] from our results.

Remark 5.1.3. We suspect that it is possible to extend the methods of this section to handle the category of proper normal varieties together with finite étale morphisms, giving slightly more than the groupoid of such schemes. This will be pursued elsewhere.

5.2. Algebraic geometry lemmas. We recall a definition and a few basic facts in modern language (see [24] for a classical exposition).

Notation 5.2.1. A *pencil of divisors* on a variety X is a dominant rational map $X \dashrightarrow C$ to a smooth proper curve. The pencil is *irrational* if the genus $g(C)$ is greater than 0. It is *linear* if $g(C) = 0$.

A pencil has members. To make this precise, suppose X is proper and $f : X \dashrightarrow C$ is a pencil. The closure of the generic fiber of f is a divisor $D \subset X \otimes \kappa(C)$, giving rise to a morphism

$$\varphi : \text{Spec } \kappa(C) \rightarrow \text{Hilb}_X.$$

Suppose $L \subset \text{Hilb}_X$ is the residue field of the image of φ ; for dimension reasons L must have transcendence degree 1 over k . Taking the closure defines a curve $\overline{C} \subset \text{Hilb}_X$ with a morphism $C \rightarrow \overline{C}$. This gives rise to a universal family $\mathcal{D}(f) \subset X \times C$ associated to the pencil f .

Equivalently, we can describe $\mathcal{D}(f) \subset X \times C$ as the scheme-theoretic closure of the graph of the rational map f . (This is equivalent to the previous definition by the separatedness of the Hilbert scheme.)

Definition 5.2.2. The fibers of the morphism $\mathcal{D}(f) \rightarrow C$ described above are the *members* of the pencil, denoted $|f|$. The intersection of all members of the pencil is the *base locus* of the pencil, denoted $\text{Bs}(f)$.

By construction there is a diagram

$$(5.2.2.1) \quad \begin{array}{ccc} \mathcal{D}(f) & \xrightarrow{\iota} & X \\ \pi \downarrow & & \\ C & & \end{array}$$

where ι is proper and birational (note that the birationality follows from the fact that by definition the locus $U \subset X$ where f is regular is contained in $\mathcal{D}(f)$).

Lemma 5.2.3. *Let $U \subset X$ be the maximal open subset over which f is defined, and let $B \subset X$ be the complement of U . Then a closed point $x \in X$ lies in B if and only if for every closed point $y \in C$ we have $x \in \mathcal{D}(f)_y$.*

Proof. This is [24, §8]. For the convenience of the reader we give a proof.

If x lies in $\mathcal{D}(f)_y \cap \mathcal{D}(f)_{y'}$ for $y \neq y'$ then certainly we must have $x \in B$. Conversely suppose that f is not defined at x . Then the fiber $\iota^{-1}(x)$ must be positive dimensional,

since X is normal, and dominates C . It follows that $\iota^{-1}(x) \cap \mathcal{D}(f)_y$ for every $y \in C(k)$, which implies the lemma. \square

Lemma 5.2.4. *Suppose X is normal. Given a pencil $f : X \dashrightarrow C$ with base locus B and with associated family of members $\mathcal{D}(f) \rightarrow C$, the canonical morphism $\mathcal{D}(f) \rightarrow X$ is an isomorphism over $X \setminus B$. Moreover, any member of the pencil gives a Cartier divisor in $X \setminus B$.*

Proof. By assumption, in the diagram (5.2.2.1) the morphism ι is a closed immersion on each fiber of π . Moreover, any point $x \in X \setminus B$ lies on exactly one member of the pencil by Lemma 5.2.3. That is, ι is finite and injective away from B ; since X is normal, this implies that ι is an isomorphism over $X \setminus B$, as claimed.

To see that every member is Cartier outside of B , note that the fibers of π are Cartier divisors (since C is smooth), and this is transported to X by ι , which is an isomorphism outside of B . \square

We give a proof of a classical result here, for lack of convenient reference.

Proposition 5.2.5. *A proper variety over an algebraically closed field has only countably many equivalence classes of irrational pencils $X \dashrightarrow Z$, where $X \dashrightarrow Z_1$ is equivalent to $X \dashrightarrow Z_2$ if they have the same members.*

Proof. First assume that X is smooth and projective. Consider an irrational pencil $f : X \dashrightarrow C$. Let $U \subset X$ be the maximal domain of definition of f . If $U \neq X$, then f extends generically across the blowup $\text{Bl}_{\{x\}} X$ so that the extension is not constant on the exceptional fiber. But then C is unirational, hence rational. Thus, f extends to a morphism $f : X \rightarrow C$. It follows (by, for example, computing the deformation space of a general fiber of f) that C gives a single irreducible component of the Hilbert scheme of X . Since the Hilbert scheme has countably many components, we see that there can be only countably many equivalence classes of irrational pencils. We also see that if $X \rightarrow C$ and $X \rightarrow C'$ are equivalent pencils, then there is an isomorphism $C \rightarrow C'$ that conjugates them. In particular, the equivalence class of the pencil is uniquely determined by the abstract subfield $k(C) \subset k(X)$.

Now suppose X is an arbitrary proper variety over k . Given an alteration $X' \rightarrow X$, we see that the set of pencils on X injects into the set of pencils on X' (since the set of subfields of $k(X)$ injects into the set of subfields of $k(X')$). On the other hand, there is a smooth projective alteration $X' \rightarrow X$. This reduces the result to the smooth projective case. \square

Lemma 5.2.6. *On a proper normal variety X , no effective divisor is algebraically equivalent to 0.*

Proof. Let $g : X' \rightarrow X$ be a projective birational morphism from a normal projective variety. If there is an effective divisor $D \subset X$ that is algebraically equivalent to 0, then there is a corresponding algebraic equivalence

$$D' + E_1 \sim E_2$$

on X' , where D' is an effective non-exceptional divisor and E_1, E_2 are effective exceptional divisors with no components in common. (This is explained in [10, Example 10.3.4].) Taking general hyperplane sections of X' , the Gysin map reduces us to the

case in which X and X' are normal surfaces. If we wish, we may assume that X' is also smooth. By [16, Section 1, page 6], we have $E_2 \cdot E_2 < 0$ unless $E_2 = 0$. But $(D' + E_1) \cdot E_2 \geq 0$. It follows that $E_2 = 0$. But then we have an effective divisor on a projective surface algebraically equivalent to 0, which is impossible, as one can see by intersecting with a general hyperplane. \square

Lemma 5.2.7. *Let X be a projective integral k -scheme, and let $Z \hookrightarrow \text{Hilb}_X$ be a connected smooth curve embedded in the Hilbert scheme corresponding to a family $\mathcal{W} \hookrightarrow X \times Z$ of closed subschemes. Assume that for a general point $z \in Z$ the fiber $W_z \subset X_{k(z)}$ is geometrically integral of some dimension strictly smaller than $\dim(X)$. Then the map*

$$g : \mathcal{W} \rightarrow X$$

is generically unramified.

Proof. Shrinking on Z if necessary, we may assume that \mathcal{W} is integral. Since \mathcal{W} is geometrically integral, the morphism $\mathcal{W} \rightarrow \text{Spec}(k)$ is generically smooth, and it suffices to show that the morphism

$$g^* \Omega_X^1 \rightarrow \Omega_{\mathcal{W}}^1$$

is generically surjective. Let $z \in Z$ be a point and let \mathcal{W}_z be the fiber of \mathcal{W} over z which is a closed subscheme

$$g_z : \mathcal{W}_z \hookrightarrow X_{k(z)}.$$

Assume that z is general so that \mathcal{W}_z is generically smooth. Let $I_z \subset \mathcal{O}_{X_{k(z)}}$ be the ideal sheaf of \mathcal{W}_z . We then have a commutative diagram

$$\begin{array}{ccccccc} g_z^* I_z & \longrightarrow & g_z^* \Omega_X^1 & \longrightarrow & \Omega_{\mathcal{W}_z}^1 & \longrightarrow & 0 \\ & & \downarrow \alpha & & \parallel & & \\ 0 & \longrightarrow & f_z^* \Omega_Z^1(z) & \longrightarrow & \Omega_{\mathcal{W}_z}^1|_{\mathcal{W}_z} & \longrightarrow & 0, \end{array}$$

where the top sequence is the conormal sequence and the bottom sequence is the exact sequence of a composition (which is exact on the left in our situation because Z is smooth and \mathcal{W} is generically smooth). It then suffices to show that the map labelled α is generically surjective, or equivalently, since Z is a curve, generically nonzero.

For this recall that the tangent space to the Hilbert scheme at the point z can be described as

$$\text{Hom}_{\mathcal{W}_z}(g_z^* I_z, \mathcal{O}_{W_z}).$$

Chasing through this identification one sees that the map

$$(5.2.7.1) \quad T_Z(z) \rightarrow T_{\text{Hilb}_X}(z) = \text{Hom}_{\mathcal{W}_z}(g_z^* I_z, \mathcal{O}_{W_z})$$

is given by sending $\partial : \Omega_Z^1(z) \rightarrow k(z)$ to the map $g_z^* I_z \rightarrow \mathcal{O}_{W_z}$ given by the composition

$$g_z^* I_z \xrightarrow{\alpha} f_z^* \Omega_Z^1(z) \xrightarrow{\partial} \mathcal{O}_{W_z}.$$

Since (5.2.7.1) is injective it follows that $\alpha \neq 0$. \square

Lemma 5.2.8. *Let X be a proper geometrically integral variety, $\{D_i\}_{i \in \mathbb{N}}$ a countable set of geometrically integral Weil divisors and $B \subsetneq X$ a closed subset. Assume that*

- (1) *the D_i are algebraically equivalent to each other and*

(2) $D_i \cap D_j \subset B$ for every $i \neq j$.

Then infinitely many of the D_i are members of a (possibly irrational) pencil of divisors $p : X \dashrightarrow Z$. If the codimension of B is at least 2 and X is geometrically normal, then all of the D_i lie in $|p|$.

Proof. It suffices to prove the result after passing to the algebraic closure of k . We begin by producing the pencil (i.e., the rational map $X \dashrightarrow C$), which we may do on any birational model of X .

Let $Z \subset \text{Hilb}_X$ be the Zariski closure of the points $[D_i]$. There is a universal family $u : U \rightarrow Z$ with canonical map $\chi : U \rightarrow X$. Since the D_i are irreducible, we see that each geometric generic fiber of u is irreducible. We wish to show that χ is a birational morphism, for then Z must be a curve and $p = u\chi^{-1}$ defines the pencil.

Let $\kappa(Z)$ be the ring of total fractions of Z ; in this case, it is the product of the function fields of the components of Z . Let $c : Y \rightarrow X$ be a proper birational morphism from a projective variety. The strict transform of the generic fiber $U_{\kappa(Z)}$ in $Y_{\kappa(Z)}$ extends to a flat family $V \rightarrow Z^\circ$ over some dense open subscheme $Z^\circ \subset Z$, with a morphism $V \rightarrow Y$ that is a closed immersion on each geometric fiber over Z° . Moreover, for each D_i , corresponding to a point $z_i \in Z^\circ$, we have that D_i is the image of V_{z_i} under the natural map $Y \rightarrow X$. Shrinking Z° and replacing the D_i by an infinite subsequence, we may assume that each V_{z_i} is irreducible, and is the strict transform of D_i . This reduces the existence of the pencil to the case where X is projective, which we now assume until noted otherwise. Shrinking Z , we may assume that all fibers of $U \rightarrow Z$ are geometrically integral.

To show that χ is generically injective, it suffices to show that the restriction of χ to a general complete intersection in X is generically injective. Note that, by generic flatness, for any such complete intersection $Q \subset X$ there is an induced rational map $\text{Hilb}_X \dashrightarrow \text{Hilb}_Q$. Since Z is quasi-compact, the fibers U_z have bounded Castelnuovo–Mumford regularity, from which it follows that for sufficiently high multiples nH of the ample class, for any complete intersection $Q \subset X$ of divisors in $|nH|$ the induced map $Z \dashrightarrow \text{Hilb}_Q$ is generically injective.

Let $S \subset X$ be such a general complete intersection surface in X and let $\tilde{S} \rightarrow S$ be a resolution of singularities. By generic flatness, there is an open subscheme $Z_0 \subset Z$ such that the family $U \times_X \tilde{S}|_{Z_0} \rightarrow Z_0$ is a flat family of divisors on \tilde{S} containing a countable dense subset $z_i \in Z_0$ satisfying the hypotheses of the Lemma. Thus, to show that $\chi : U \rightarrow X$ is generically injective, it suffices to show the result under the additional assumption that X is a smooth projective surface. There are two cases to consider.

First, suppose there are i and j such that $D_i \cap D_j = \emptyset$. By the invariance of intersection number, we have that $U_s \cdot U_t = 0$ for all s and t , whence $U_s \cap U_t = \emptyset$ unless they are equal. But Z injects into Hilb_X , so we conclude that $U_s \cap U_t = \emptyset$. This shows that $U \rightarrow X$ is generically injective.

On the other hand, suppose $D_i \cap D_j \neq \emptyset$, so that $D_i \cdot D_j > 0$. We conclude that for any $s, t \in Z$ we have $U_s \cap U_t \neq \emptyset$. In particular, we have that the natural map

$$q : U \times_X U \rightarrow Z \times Z$$

is dominant. If the morphism $U \rightarrow X$ has general geometric fiber with at least 2 points then $w : U \times_X U \rightarrow X$ is also dominant. But then $q(w^{-1}(X \setminus B))$ contains a dense open of $Z \times Z$, whence it contains a point $([D_i], [D_j])$ for some $i \neq j$. We conclude that $D_i \cap D_j \cap (X \setminus B) \neq \emptyset$, contradiction.

Returning to the original situation (proper, possibly non-projective X of arbitrary dimension), we conclude that χ is generically injective, which also implies that Z has dimension 1. By Lemma 5.2.7, we also know that χ is generically unramified, whence we conclude that χ is birational. In particular, χ defines a pencil $|D| : X \dashrightarrow Z$, and we conclude that infinitely many of the D_i are members of a pencil $|D|$ parameterized by Z , as claimed.

Now suppose that X is geometrically normal and B has codimension at least 2. We claim that in fact all D_i lie in the pencil. To see this, note that every irreducible divisor $E \subset X$ is either an irreducible component of a member of the pencil $E \subset D_z$ or $D_z \cap (E \setminus B) \neq \emptyset$ for all but finitely many members. Indeed, since the D_z cover X , their intersections with E cover E . It follows that only a proper closed subset of Z parameterizes D_z such that $D_z \cap E \subset B$.

If D_i were not an irreducible component of some member in the pencil $|D|$ then the above observation would violate condition (2) above. On the other hand, if there is a member D_z such that $D_z - D_i$ is effective, then $D_z - D_i$ would be an effective divisor algebraically equivalent to 0, contradicting Lemma 5.2.6. \square

Definition 5.2.9. Let X be a variety over a field k . Let $C \subset X$ be a 1-dimensional subscheme and $D = \sum a_i D_i \subset X$ a divisor, written as a weighted sum of prime divisors. We define the *naïve intersection number* as

$$\#(C \cap D)_X := \begin{cases} -\infty & \text{if } C \subset D \\ \sum_i a_i \left(\sum_{x \in C \cap D_i} [k(x) : k] \right) & \text{otherwise.} \end{cases}$$

We drop the subscript X if it is clear from the context. If C is proper and D is a Cartier divisor in a neighborhood of C then $\#(C \cap D) \leq (C \cdot D)$. If k is algebraically closed then $\#(C \cap D)$ is the weighted sum of the numbers of points in the sets $C(k) \cap D_i(k)$.

We call a curve $C \subset X$ *t-ample* if $C \cap D \neq \emptyset$ for every divisor $D \subset X$. If X is projective then every complete intersection of ample divisors is t-ample. If X is proper then the image of a t-ample curve under any projective modification $X' \rightarrow X$ is t-ample on X .

Let $p : X \dashrightarrow Z$ be a (possibly irrational) pencil of divisors and $B \subset X$ a closed subset containing the base locus of p . Following Matsusaka [14] we define the *variable intersection number*

$$\#[C \cap p]_{X \setminus B} := \max_{D \in |D|} \#(C \cap D)_{X \setminus B}.$$

Situation 5.2.10. Suppose k is an algebraically closed field of characteristic 0 and X is a proper normal k -variety. Fix a (possibly irrational) pencil $f : X \dashrightarrow Z$ and a subset

$B \subset X$ containing the base locus of f . Let

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \\ \pi \downarrow & & \\ Z & & \end{array}$$

be the family of members of f .

Notation 5.2.11. Given a set S of cycles on a scheme Y , let $\text{Comp } S$ denote the set of all irreducible components of all elements of S . Let $\text{ConnComp } S$ denote the set of connected components of the elements of S .

We will often use $\text{Comp } |f|$, the set of irreducible components of the members of a pencil.

Lemma 5.2.12. *Assume we are in Situation 5.2.10. Let $C \subset X$ be a weakly ample curve so that no component of C lies entirely in B , and let $C' \subset Y$ be the strict transform of C .*

(1) *For every $z \in Z$ we have*

$$\#(C \cap \iota(Y_z))_{X \setminus B} \leq (C' \cdot Y_z) = \#[C \cap f]_{X \setminus B}.$$

(2) *For all but finitely many $z \in Z(k)$ we have that*

$$\#[C \cap f]_B = \#(C \cap \iota(Y_z))_{X \setminus B}.$$

Proof. Since C is t -ample, the morphism $\pi|_{C'} : C' \rightarrow Z$ is finite. Since k has characteristic 0, $\pi|_{C'}$ is separable. Since no component of C is contained in B , we have that

$$\#[C \cap f]_{X \setminus B} = \#[C' \cap f\iota]_X.$$

That is, the intersection $C' \cap Y_z$ for general z is disjoint from the preimage of B in Y , which is a subset of codimension 1 intersecting C' properly. On the other hand, since C' is separable over Z , the general naïve intersection equals the degree of C' over Z . In particular, for every $z \in Z(k)$ we have that

$$\#[C' \cap f\iota]_X = (C' \cdot Y_z).$$

This proves the result. □

Lemma 5.2.13. *In Situation 5.2.10, suppose Y_z is a member of $|f|$ and $E \leq Y_z$ is an effective divisor. Then there exists i such that $E + D_i \leq Y_z$ if and only if there exists a subset $B \subset X$ of codimension at least 2, containing the base locus of f , and an integral divisor $A \subset X$ such that for every t -ample curve $C \not\subset B$ we have*

$$\#(C \cap E)_{X \setminus B} + \#(C \cap A)_{X \setminus B} \leq \#[C \cap f]_{X \setminus B}.$$

Proof. Let B be the union of the base locus of f , the singular locus of X , and the image of the singular locus of Y . Let $\tilde{Y} \rightarrow Y$ be a resolution of singularities. Since B is assumed to contain the singular locus of X , we have that $\tilde{Y} \rightarrow Y$ is an isomorphism outside of B . Given a component $Q \subset Y_z$, we have that $Q_{\tilde{Y}} = \tilde{Q} + Q'$, where \tilde{Q} is

the strict transform and Q' is a divisor supported over B . For any curve $C \subset X$ not contained in B , with strict transform $\tilde{C} \subset \tilde{Y}$, we have an inequality

$$\#(C \cap Q)_{X \setminus B} \leq \tilde{C} \cdot \tilde{Q}.$$

Moreover, if we write $Y_z = \sum a_i D_i$, we have (in the same notation) that

$$\sum a_i (\tilde{C} \cdot \tilde{D}_i) \leq (\tilde{C} \cdot \tilde{Y}_z) = (C \cdot Y_z) = \#[C \cap f]_{X \setminus B},$$

where the first inequality follows from the fact that \tilde{Y}_z is equal to $\sum a_i \tilde{D}_i$ plus an effective divisor not containing any components of \tilde{C} , the middle equality is by the projection formula, and the last equality is by Lemma 5.2.12.

If $E = \sum b_i D_i < Y_z$, then pick $A \in \text{Comp}|D|$ such that $E + A \leq Y_z$. That is, let $A = D_j$ for some j such that $b_j + 1 \leq a_j$. For any proper irreducible t-ample curve C not contained in B we then have

$$\#(C \cap E)_{X \setminus B} + \#(C \cap A)_{X \setminus B} \leq \sum b_i (\tilde{C} \cdot \tilde{D}_i) + (\tilde{C} \cdot \tilde{D}_j) \leq \sum a_i (\tilde{C} \cdot \tilde{D}_i) \leq \#[C \cap f]_{X \setminus B},$$

and thus the condition is necessary.

To complete the proof we need to show that if $E = Y_z$ then for every subset B of X of codimension at least 2 and every integral divisor $A \subset X$, there is a t-ample curve $C \not\subset B$ such that

$$(5.2.13.1) \quad \#(C \cap E)_{X \setminus B} + \#(C \cap A)_{X \setminus B} > \#[C \cap f]_{X \setminus B}.$$

Choose a proper, birational morphism $p : X' \rightarrow X$ such that X' is normal and projective, and the pencil f extends to a morphism $f' : X' \rightarrow Z$. Let $S' \subset X'$ be a general complete intersection of $\dim X - 2$ very ample divisors. Then S' is normal and f' restricts to a morphism $f' : S' \rightarrow Z$. The support of the preimage of B in S' is the union of a finite set of points P , a reduced f' -horizontal divisor B^h , and a reduced f' -vertical divisor B^v . Since B has codimension 2, we see that B^v does not contain any fibers of f' .

Resolving the singularities of S' and using the fact that the image of a t-ample curve is t-ample, we may assume that S' is smooth over k . Note that for a general ample curve $C' \subset S'$, we have that

$$\#(p(C') \cap E)_{X \setminus B} \geq (C' \cdot E_{S'}) - (C' \cdot B^v).$$

Indeed, for general C' there is no intersection with P and the intersection with B^h occurs outside E , so that the second term in the difference on the right side thus puts an upper bound on the cardinality of $C'(k) \cap E(k) \cap B(k)$.

Since $E = Y_z$, so that

$$(C' \cdot E_{S'}) = (p(C') \cdot Y_z) = \#[p(C') \cap f]_{X \setminus B}$$

for any B , we see that to achieve the desired inequality for the curve $C = p(C')$, it suffices to find an ample curve $C' \subset S'$, not contained in $B|_{S'}$, such that

$$(C' \cdot E_{S'}) + (C' \cdot A_{S'}) > (C' \cdot E_{S'}) + (C' \cdot B^v),$$

which is equivalent to

$$(5.2.13.2) \quad (C' \cdot A_{S'}) > (C' \cdot B^v).$$

Suppose $H \subset S'$ is an ample divisor. Since B^v is vertical and does not contain a fiber, its intersection matrix is negative definite [6, Corollary 2.6]. Thus, there is an effective \mathbf{Q} -divisor N supported on B^v such that $H + N$ is numerically trivial on B^v . Moreover, for any irreducible curve C not contained in $\text{Supp } B^v$, we have $(H + N) \cdot C \geq H \cdot C$. Now

$$(1 + \epsilon)H + N = \epsilon H + (H + N)$$

is ample, very small on B^v , and bigger than H on every other divisor. Let C' be a general member of a very ample multiple of $(1 + \epsilon)H + N$ and let $C := p(C')$ be the image in X . It is straightforward that for sufficiently small ϵ we have the inequality (5.2.13.2), and this implies (5.2.13.1), as desired. \square

Remark 5.2.14. Note that Lemma 5.2.13 is false in characteristic p . We give an example in the context of Section 5.5; we will retain the notation from that section in this remark. Let Z be the divisor $f(x, y, z) = 0$ and let Y be a general curve in \mathbf{P}^2 that meets Z transversely. There is an associated pencil

$$\langle Z, Y \rangle : \mathbf{P}^2 \dashrightarrow \mathbf{P}^1$$

in the linear system $|\mathcal{O}(p)|$. This induces a pencil

$$\langle \tilde{Z}, \tilde{Y} \rangle : X_f \dashrightarrow \mathbf{P}^1$$

associated to the pullback divisors $\tilde{Z}, \tilde{Y} \subset X_f$. Note that the divisor \tilde{Z} is not reduced, since the divisor Z becomes divisible by p in D_f (hence has a component in X_f that is divisible by p).

On the other hand, there is a finite purely inseparable morphism $X_f \rightarrow Q$ of smooth surfaces described in Section 5.5. The restriction $Z|_Q$ is reduced (since Q is a blow up of \mathbf{P}^2 at a finite set disjoint from Z). Since the underlying topological spaces of X_f and Q are the same, the sets of curves are in bijection. We see that a general curve meeting Y and Z transversely in Q corresponds to a curve with $\#(C \cap Y) = \#(C \cap Z)$. The corresponding curve in X_f therefore has the same property. We conclude that

$$\#(C \cap \tilde{Y}) = \#(C \cap \tilde{Z}) = C \cdot \tilde{Y} = C \cdot \tilde{Z},$$

but this does not imply that members of the pencil are reduced. Thus, the naïve intersection multiplicity cannot be used to identify multiplicities of components of fibers. (This arises from the fact that the general naïve intersection need not equal the general intersection number because maps of curves need not be separable. This is the crux of the characteristic 0 hypothesis.)

Corollary 5.2.15. *In Situation 5.2.10, suppose E is a reduced divisor on X whose components lie in $\text{Comp } |f|$ and such that $E \setminus \text{Bs } |f|$ is connected. Then E is a member of $|f|$ if and only if for every subset $B \subset X$ of codimension 2 containing $\text{Bs } |f|$ and every integral divisor $A \subset X$, there is a t -ample curve $C \not\subset B$ such that*

$$\#(C \cap E)_{X \setminus B} + \#(C \cap A)_{X \setminus B} > \#[C \cap f]_{X \setminus B}.$$

Proof. This follows immediately from Lemma 5.2.13, once we note that E must lie in Y_z for some $z \in Z$, since $E \setminus \text{Bs } |f|$ is connected. \square

5.3. Zariski topology lemmas. In this section we assume that the base field k is algebraically closed and uncountable.

Definition 5.3.1. Let X be a normal, proper variety. A t -pencil on X is a set of reduced Weil divisors $\{D_c\}$ such that

- (1) the union of the D_c contains all closed points of X ,
- (2) there is a subset $B \subset X$ of codimension ≥ 2 such that $D_{c_1} \cap D_{c_2} \subset B$ for every $c_1 \neq c_2$, and
- (3) all but finitely many of the D_c are irreducible.

Two t -pencils are *distinct* if they share only finitely many members.

Lemma 5.3.2. *Suppose X is a proper normal variety over a field k . Let Hilb_X^1 be the union of the components of the Hilbert scheme that contain points corresponding to integral divisors. Then Hilb_X^1 has only countably many irreducible components.*

Proof. Let $\pi : X' \rightarrow X$ be a birational projective morphism from a normal projective variety. In particular, π is an isomorphism in codimension 1. Define a rational map

$$p : (\text{Hilb}_{X'}^1)_{\text{red}} \dashrightarrow \text{Hilb}_X^1$$

by pushforward. More precisely, the universal ideal sheaf $\mathcal{F} \subset \mathcal{O}_{X' \times \text{Hilb}_{X'}^1}$ defines an ideal sheaf

$$\pi_* \mathcal{F} \cap \mathcal{O}_{X \times \text{Hilb}_{X'}^1} \subset \mathcal{O}_{X \times \text{Hilb}_X^1}.$$

By generic flatness, this defines a rational map from the reduced structure of $\text{Hilb}_{X'}^1$ to Hilb_X^1 . On the other hand, strict transform defines a rational map

$$q : (\text{Hilb}_{X'}^1)_{\text{red}} \dashrightarrow \text{Hilb}_{X'}^1.$$

It is not hard to see that $pq = \text{id}$ as rational maps on each irreducible component of Hilb_X^1 . In particular, the set of irreducible components of Hilb_X^1 is identified with a subset of the set of irreducible components of $\text{Hilb}_{X'}^1$. Since X' is projective, $\text{Hilb}_{X'}^1$ has countably many components. The result follows. \square

Lemma 5.3.3. *Let X be a normal, proper variety and $\{D_c\}$ a t -pencil. Assume that the base field is uncountable and algebraically closed. Then there is a (possibly irrational) pencil of divisors $|f|$ such that*

$$\text{ConnComp}\{D_c \setminus \text{Bs } |f|\} = \text{ConnComp}\{E \setminus \text{Bs } |f| : E \in |f|\}.$$

In particular, if every $D_c \setminus \text{Bs } |f|$ is connected and every $E \setminus \text{Bs } |f|$ is connected for $E \in |f|$, then the D_c are precisely the reduced structures on the $E \in |f|$.

Proof. By Lemma 5.3.2, countably many components of the Hilbert scheme of X contain all of the points $[D_c]$. Thus, there are uncountably many D_c that are algebraically equivalent. They determine a pencil $|D|$ by Lemma 5.2.8. Each remaining divisor must lie in a member of $|D|$ (by connectedness). Since the D_c cover X , we conclude that each connected component appears. \square

As the following example shows, over a countable field t -pencils need not come from algebraic pencils. (This more or less must be the case by Lemma 4.4.1. Here is an explicit example.)

Example 5.3.4 (t-pencils over countable fields). Let X be a normal, projective variety of dimension ≥ 2 over an infinite field K and L a very ample line bundle on X .

Pick any $s_1 \in H^0(X, L^{m_1})$ and $s_2 \in H^0(X, L^{m_2})$. Assume that we already have $s_i \in H^0(X, L^{m_i})$ for $i = 1, \dots, r$ such that $\text{Supp}(s_i = s_j = 0)$ is independent of $1 \leq i < j \leq r$.

Set $M = \prod_i m_i$,

$$S_{r+1} := \left(\prod_i s_i^{M/m_i} \right) \left(\sum_i s_i^{-M/m_i} \right) \text{ and } T_{r+1} := \prod_i s_i.$$

and choose

$$s_{r+1} = S_{r+1} + g \cdot T_{r+1}$$

for a general $g \in H^0(X, L^{n_r})$ where $n_r = M(r-1) - \sum_i m_i$. Then $(s_{r+1} = 0)$ is irreducible and $\text{Supp}(s_i = s_j = 0)$ is independent of $1 \leq i < j \leq r+1$.

If K is countable then we can order the points of X as x_1, x_2, \dots and we can choose the s_i such that $\prod_{i=1}^r s_i$ vanishes on x_1, \dots, x_r for every r . Then the resulting $D_i := (s_i = 0)$ is a t-pencil that does not correspond to any actual pencil.

Definition 5.3.5. Let X be a normal proper variety and $\{D_c\}$ a t-pencil. We say that D_c is a *true member* of $\{D_c\}$ if the condition of Corollary 5.2.15 holds.

Definition 5.3.6. Two reduced divisors D and E are *directly linearly t-equivalent* if there is an uncountable collection of divisors $F_i, i \in I$ such that

- (1) for each F_i , there is a t-pencil P_i containing D and F_i as true members and a t-pencil Q_i containing E and F_i as true members;
- (2) the pencils P_i are pairwise distinct, and the pencils Q_i are pairwise distinct.

The equivalence relation generated by direct linear t-equivalence will be called *linear t-equivalence* and denoted by \sim_t .

Lemma 5.3.7. *If X is a proper normal variety over an uncountable algebraically closed field of characteristic 0, then linearly t-equivalent divisors are linearly equivalent.*

Proof. It suffices to show that two directly linear t-equivalent divisors D and E are linearly equivalent. By Lemma 5.3.3, there are pencils connecting D to F_i and E to F_i . Since X has only countably many irrational pencils (by Proposition 5.2.5), we conclude that most of the pencils are linear, showing that D and E are linearly equivalent. \square

Lemma 5.3.8. *If X is a proper normal variety over an uncountable algebraically closed field of characteristic 0 then linear equivalence of divisors is the same as linear t-equivalence.*

Proof. By Lemma 5.3.7, it suffices to show that linearly equivalent divisors are linearly t-equivalent.

Choose a proper birational morphism $p : X' \rightarrow X$ where X' is normal and projective. Let $B \subset X$ be the smallest closed subset such that p is an isomorphism over $X \setminus B$, and let $B' = p^{-1}(B)$. Let $|H'|$ be an ample linear system on X' and set $|H| := p_*|H'|$. This is well-defined since proper pushforward of algebraic cycles preserves rational equivalence. Note that for any $m > 0$ we have that $p_*mH' = mH$. Similarly, for any divisor $A \subset X$ with strict transform A' in X' , we have that $|A + mH| = p_*|A' + mH'|$.

In particular, for all sufficiently large m , we have that $|A + mH|$ is the pushforward of a very ample linear system from X' .

Suppose A is a connected reduced divisor on X with strict transform $A' \subset X'$. (Note that A' need not be connected.) Choose m so that $A' + mH'$ is very ample. Let A'_0, \dots, A'_m be the connected components of $A' \setminus B'$. A general member H'_m of $|mH'|$ is integral and meets A'_i for all i , so that $(H'_m + A') \setminus B'$ is connected.

Write Q_0, \dots, Q_N for the irreducible components of $(H'_m + A') \setminus B'$. A general member A'_m of $|A' + mH'|$ will be integral and not contain any irreducible component of any intersection $Q_i \cap Q_j$. It follows that $(H'_m + A') \setminus A'_m$ remains connected.

It follows that for general $H_m \in |mH|$ and $A_m \in |A + mH|$, we have that $A + H_m$ and A_m are directly linearly t-equivalent, since we have just shown that for general choices of H_m and A_m , removing the base locus of $\langle A + H_m, A_m \rangle$ does not disconnect $A + H_m$ or A_m . Since this holds for general A_m given a fixed general H_m , we see that for general A_m and H_m we have a linear t-equivalence $A + H_m \sim_t A_m$.

Suppose given an effective divisor $\sum_{i=1}^n a_i A_i$ on X . Arguing as above we see that for large enough m and a general member $H_m \in |mH|$, there is a direct linear t-equivalence

$$A_1 + H_m \sim_t A_0$$

with A_0 an integral divisor distinct from A_1, \dots, A_n . This yields a linear t-equivalence

$$H_m + \sum_{i=1}^n a_i A_i \sim_t A_0 + (a_1 - 1)A_1 + \sum_{i=2}^n a_i A_i.$$

By induction on $\sum a_i$, we see that for all sufficiently large m , for any $d > \sum a_i$ and general members $H_m^{(1)}, \dots, H_m^{(d)}$, there is a linear t-equivalence

$$H_m^{(1)} + \dots + H_m^{(d)} + \sum a_i A_i \sim_t A_\infty$$

for some integral divisor A_∞ .

Given two linearly equivalent effective divisors $A = \sum_{i=1}^n a_i A_i$ and $B = \sum_{j=1}^m b_j B_j$, choose $d > \max\{\sum a_i, \sum b_j\}$. By the above argument, we get linear t-equivalences

$$H_m^{(1)} + \dots + H_m^{(d)} + A \sim_t A_\infty$$

and

$$H_m^{(1)} + \dots + H_m^{(d)} + B \sim_t B_\infty$$

for two linearly equivalent integral divisors A_∞ and B_∞ . Moreover, choosing d large enough, we may assume that the linear system containing A_∞ and B_∞ is the pushforward of a very ample linear system from X' . Choosing a general member of the linear system $|A_\infty|$ then produces uncountably many pencils as in Definition 5.3.6.

We thus get a chain of linear t-equivalences

$$H^{(1)} + \dots + H^{(d)} + A \sim_t A_\infty \sim_t B_\infty \sim_t H^{(1)} + \dots + H^{(d)} + B,$$

showing that A and B are linearly t-equivalent, as claimed. \square

5.4. A topological Gabriel theorem. In this section, we prove the following analogues of Gabriel's reconstruction theorem.

Theorem 5.4.1. *Let X be a proper normal variety of dimension at least 2 over an uncountable algebraically closed field k . Let \mathcal{C} be one of the following categories:*

- (1) *the category of constructible abelian étale sheaves on X ;*
- (2) *the category of constructible étale sheaves of \mathbf{F}_ℓ -modules on X ;*
- (3) *the category of constructible étale \mathbf{Q}_ℓ -sheaves on X ;*
- (4) *the category of constructible étale $\overline{\mathbf{Q}}_\ell$ -sheaves on X .*

Then X is uniquely determined as an abstract scheme up to isomorphism by the category \mathcal{C} up to equivalence. More precisely, given two such varieties X and Y , any equivalence $\mathcal{C}_X \rightarrow \mathcal{C}_Y$ induces an isomorphism $X \rightarrow Y$ of schemes.

Let A be one of the rings \mathbf{F}_ℓ or a subfield $\mathbf{Q}_\ell \subset A \subset \overline{\mathbf{Q}}_\ell$ and let \mathcal{C} be the category of constructible étale A -modules on X .

Lemma 5.4.2. *A constructible A -module M is isomorphic to a sheaf of the form $\iota_* A$ for some $\iota : \text{Spec } k \rightarrow X$ if and only if it is a (non-zero) simple object of \mathcal{C} .*

Proof. Since M is constructible, its support is a constructible subset of X . If it contains two closed points x and y , then M contains the proper submodule $\iota_! M_x$, where $\iota : x \rightarrow X$ is the inclusion map. We conclude that M is supported at a single closed point $x \in X$, so that $M = \iota_* M'$ for some M' . Since M is simple, M' must be a simple vector space, so it must have dimension 1. \square

Definition 5.4.3. Irreducible objects and equivalence.

- An object $F \in \mathcal{C}$ is *irreducible* if
 - (1) for every simple object $s \in \mathcal{C}$ we have $\dim_A \text{Hom}(F, s) \leq 1$, and
 - (2) any pair of subobjects $F', F'' \subset F$ have non-zero intersection.
- The *support* of an irreducible object F is the set $\text{Supp}(F)$ of simple quotients of F .
- Two irreducible objects F and F' are *equivalent* if $\text{Supp}(F) = \text{Supp}(F')$.
- An irreducible object F is a *partial closure* of an irreducible object G if there are non-zero subobjects $F' \subset F$ and $G' \subset G$ such that F' is equivalent to G' .
- An irreducible object F is *closed* if any partial closure of F has the same support.
- An irreducible object F is a *closure* of an irreducible object F' if F is a closed partial closure of F' .

We can associate two subsets of X to an irreducible object F : the support of the sheaf F and the set $\text{Supp}(F)$ as defined above, which is identified with a subset of X via Lemma 5.4.2. It is not hard to see that the support of the sheaf F is the Zariski closure of the set of closed points, which is $\text{Supp}(F)$. We will safely conflate these supports in what follows.

Lemma 5.4.4. *The following hold for irreducible objects.*

- (1) *If F is a closed irreducible object of \mathcal{C} then the support of F is a closed irreducible subset of X .*

- (2) *The set of irreducible closed subsets of X is in bijection with equivalence classes of closed irreducible objects of \mathcal{C} .*
- (3) *An irreducible closed subset $Y \subset X$ lies in an irreducible closed subset $Z \subset X$ if and only if there is a closed irreducible sheaf F with $\text{Supp}(F) = Z(k)$, a closed irreducible sheaf F' with $\text{Supp}(F') = Y(k)$, and a surjection $F \rightarrow F'$.*

Proof. If the support of F is not irreducible then there are two open subsets $U, V \subset \text{Supp}(F)$ such that $U \cap V = \emptyset$. But then $(j_U)_!F_U$ and $(j_V)_!F_V$ are two non-zero subsheaves with trivial intersection. Suppose the support of F is not closed. Consider the inclusion $\text{Supp}(F) \hookrightarrow \overline{\text{Supp}(F)}$. Since F is constructible, there is an open subscheme $U \subset \text{Supp}(F) \subset \overline{\text{Supp}(F)}$. The constant sheaf on $\overline{\text{Supp}(F)}$ is then a partial closure of F , since $j_!F|_U$ is equivalent to $i_!A$, where $j : U \rightarrow \text{Supp}(F)$ and $i : U \rightarrow \overline{\text{Supp}(F)}$ are the natural open immersions.

The second statement follows from the first statement and the fact that constant sheaves define all irreducible closed subsets.

The last statement follows from the fact that for any surjection $F \rightarrow F'$ we have $\text{Supp}(F') \subset \text{Supp}(F)$, combined with the fact that, if $i : Y \rightarrow Z$ is a closed immersion, the natural map $A_Z \rightarrow i_*A_Y$ is a surjection of irreducible sheaves. \square

Proposition 5.4.5. *The Zariski topological space X is uniquely determined by the category \mathcal{C} .*

Proof. It suffices to reconstruct the Zariski topology on the set of closed points $X(k)$. First note that we can describe the set itself as the set of isomorphism classes of simple objects of \mathcal{C} , by Lemma 5.4.2. Given a sheaf F , we can thus describe the support $\text{Supp}(F) \subset X(k)$. By Lemma 5.4.4, we can reconstruct the set of irreducible closed subsets $Z \subset X(k)$. This suffices to completely determine the topology, since closed subsets are precisely finite unions of irreducible closed subsets. \square

Proof of Theorem 5.4.1. The result follows immediately from Proposition 5.4.5 and Theorem 5.1.2. \square

Remark 5.4.6. It is natural to wonder if there are topological analogues of Balmer's monoidal reconstruction theorem [1], or the theory of Fourier–Mukai transforms. These ideas will be pursued elsewhere.

5.5. Counterexamples with weaker hypotheses. Here we give some examples showing that the assumption that k is algebraically closed of characteristic 0 is necessary for Theorem 5.1.1 and Theorem 5.1.2 to be true as stated. We also briefly speculate about appropriate replacements in positive characteristic.

Example 5.5.1. Let E be an elliptic curve over a field k of characteristic $p > 0$. Then the morphism

$$F_{E/k} \times \text{id} : E \times_k E \rightarrow E^{(p)} \times_k E$$

is a homeomorphism which is not induced by an isomorphism of schemes

$$E \times_k E \rightarrow E^{(p)} \times_k E.$$

Indeed such an isomorphism would have to respect the product structure implying that $E \simeq E^{(p)}$ over k . So we get examples where Theorem 5.1.2 fails by choosing E such that E is not isomorphic to $E^{(p)}$ over k .

Example 5.5.2. For the second example, we assume that k has characteristic at least 5. Given a homogeneous polynomial $f(x, y, z)$ of degree p , let $D_f \subset \mathbf{P}^3$ denote the divisor given by the equation $w^p - f(x, y, z)$. The projection map

$$(x, y, z, w) \mapsto (x, y, z) : \mathbf{P}^3 \dashrightarrow \mathbf{P}^2$$

defines a morphism $\pi : D_f \rightarrow \mathbf{P}^2$ which realizes D_f as obtained from the p -th root construction

$$D_f = \mathrm{Spec}_{\mathcal{O}_{\mathbf{P}^2}} \mathcal{O} \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-p+1),$$

with the multiplication structure defined by the inclusion $\mathcal{O}(-p) \rightarrow \mathcal{O}$ associated to the divisor $f(x, y, z) = 0$. In particular, D_f is finite flat over \mathbf{P}^2 . Since a general such polynomial f (for example, $f(x, y, z) = x^{p-1}y + y^{p-1}w + w^{p-1}x$) has a finite set of critical points, we see that for such f the scheme D_f is a normal surface. By adjunction, we have that $K_{D_f} \cong \mathcal{O}_{D_f}(p-3)$ is big. We will write $X_f \rightarrow D_f$ for a minimal resolution of X_f ; the preceding considerations show that X_f is a smooth surface of general type.

Since π is purely inseparable, the map π is a homeomorphism, but not an isomorphism, as D_f is not smooth. This gives counterexamples to Theorem 5.1.2 in positive characteristic.

Note that the example described here comes from a purely inseparable homeomorphism $X \rightarrow P$. In particular, X and P have isomorphic perfections. This leads naturally to the following question:

Question 5.5.3. Suppose X and Y are proper normal varieties of dimension at least 2 over uncountable algebraically closed fields with perfections X^{perf} and Y^{perf} . Is the map

$$\mathrm{Isom}(X^{\mathrm{perf}}, Y^{\mathrm{perf}}) \rightarrow \mathrm{Isom}(|X|, |Y|)$$

a bijection?

In the spirit of Grothendieck and Voevodsky, it is also natural to ask the following question.

Question 5.5.4. Is the perfection of a normal scheme of positive dimension over an uncountable algebraically closed field uniquely determined by its proétale topos?

We also note the following result of Kollár and Mangolte [?].

Theorem 5.5.5 (Kollár–Mangolte). *Let S be a smooth, projective rational surface over \mathbf{R} . Then the group of algebraic automorphisms (defined over \mathbf{R}) is dense in the group of diffeomorphisms of $S(\mathbf{R})$.*

Here an algebraic automorphism is an automorphism of $S(\mathbf{R})$ induced by a rational map $S \dashrightarrow S$ whose domain of definition includes all of the real points $S(\mathbf{R})$. Thus, for instance, there are many Cremona transformations with purely imaginary base-points that induce Zariski homeomorphisms on the Zariski dense of real points. These cannot extend to isomorphisms of schemes, even after change of complex structure. Thus, the fullness of the functor fails if one tries to restrict attention to real points, even for rational surfaces.

REFERENCES

- [1] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.
- [2] Manuel Blum, Michael Luby, and Ronitt Rubinfeld. Self-testing/correcting with applications to numerical problems. *JCSS*, 47:549–595, 1993.
- [3] Fedor Bogomolov, Mikhail Korotiaev, and Yuri Tschinkel. A Torelli theorem for curves over finite fields. *Pure Appl. Math. Q.*, 6(1, Special Issue: In honor of John Tate. Part 2):245–294, 2010.
- [4] Fedor Bogomolov and Yuri Tschinkel. Reconstruction of function fields. *Geom. Funct. Anal.*, 18(2):400–462, 2008.
- [5] Alina Bucur and Kiran S. Kedlaya. The probability that a complete intersection is smooth. *J. Théor. Nombres Bordeaux*, 24(3):541–556, 2012.
- [6] Lucian Bădescu. *Algebraic surfaces*. Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.
- [7] Anna Cadoret and Alena Pirutka. Reconstructing function fields from Milnor K -theory, 2018, arXiv:1808.04944.
- [8] François Charles and Bjorn Poonen. Bertini irreducibility theorems over finite fields. *J. Amer. Math. Soc.*, 29(1):81–94, 2016.
- [9] H. Flenner, L. O’Carroll, and W. Vogel. *Joins and intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.
- [10] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [11] Ehud Hrushovski and Boris Zilber. Zariski geometries. *J. Amer. Math. Soc.*, 9(1):1–56, 1996.
- [12] Nathan Jacobson. *Basic algebra. I*. W. H. Freeman and Company, New York, second edition, 1985.
- [13] Jean-Pierre Jouanolou. *Théorèmes de Bertini et applications*, volume 42 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1983.
- [14] Teruhisa Matsusaka. Polarized varieties with a given Hilbert polynomial. *Amer. J. Math.*, 94:1027–1077, 1972.
- [15] R. J. Mihalik. *Projective geometry and algebraic structures*. Academic Press, New York-London, 1972.
- [16] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Inst. Hautes Études Sci. Publ. Math.*, (9):5–22, 1961.
- [17] Bjorn Poonen. Bertini theorems over finite fields. *Ann. of Math. (2)*, 160(3):1099–1127, 2004.
- [18] Stefan Schröer. The strong Franchetta conjecture in arbitrary characteristics. *Internat. J. Math.*, 14(4):371–396, 2003.
- [19] Ernest E. Shult. *Points and lines*. Universitext. Springer, Heidelberg, 2011. Characterizing the classical geometries.
- [20] Adam Topaz. Reconstructing function fields from rational quotients of mod- ℓ Galois groups. *Math. Ann.*, 366(1-2):337–385, 2016.
- [21] Adam Topaz. A Torelli theorem for higher-dimensional function fields, 2017, arXiv:1705.01084.
- [22] V. A. Voevodskiĭ. Étale topologies of schemes over fields of finite type over \mathbb{Q} . *Izv. Akad. Nauk SSSR Ser. Mat.*, 54(6):1155–1167, 1990.
- [23] Roger Wiegand and William Krauter. Projective surfaces over a finite field. *Proc. Amer. Math. Soc.*, 83(2):233–237, 1981.
- [24] Oscar Zariski. Pencils on an algebraic variety and a new proof of a theorem of Bertini. *Trans. Amer. Math. Soc.*, 50:48–70, 1941.
- [25] Boris Zilber. A curve and its abstract Jacobian. *Int. Math. Res. Not. IMRN*, 2014(5):1425–1439, 2014.