Cohomology of (higher) classifying stacks

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(Joint work in process with D. Kubrak & S. Mondal.)

§0 Heuriastics

$G = $ gp. then we are interested in $H^*(BG)$

b/c

$$\begin{array}{ccc} X & & Y \\ \downarrow {\scriptstyle G\text{-torsor}} & \longleftrightarrow & \downarrow \\ Y & & BG \end{array} \qquad \begin{array}{c} H^*(Y) \\ \uparrow \\ H^*(BG) \end{array}$$

So $H^*(BG) = $ "char classes for principle G-bundles".

e.g. $G = \mathbb{G}_m$ (or $\mathbb{C}^* \sim S^1 = K(\mathbb{Z};1)$), then

$$B\mathbb{G}_m \sim K(\mathbb{Z};2) = \mathbb{CP}^\infty.$$

$H^*(B\mathbb{G}_m) = $ poly ring gen'd by a deg 2 class, known as "1st Chern class".

§1 de Rham stories

Fix p prime. $k = $ perf field of char p

Thm (Antiean-Bhatt-Mathew)

$$H^*_{dR}(B\mu_{p,k} \text{ or } B\alpha_{p,k}) = k[\alpha,\beta]/(\alpha^2)$$

with $|\alpha| = 1$, $|\beta| = 2$.

Rmk $H^*_{dR}(B(\mathbb{Z}/p)_k) \cong H^*_{sing}(K(\mathbb{Z}/p;1), k)$

$\cong H^*_{gp\,coh}(\mathbb{Z}/p, k)$.

Thm (ABM) H-dR ss deg for $B\mu_p$, but not for $B\alpha_p$.

Q What happens with $H^*_{dR}(BG/k)$ for fin comm gp sch G/k?

Recall Thm (Cartier-Dieudonné)

$$\{G/k\} \longleftrightarrow \{\text{"Dieudonné mods for } G\text{"}\}$$

$$G \longleftrightarrow \mathbb{D}(G).$$

fin length $W(k)$-mod.

Q: How to express $H^*_{dR}(BG/k)$ in terms of $\mathbb{D}(G)$?

Thm (KLM) $p>2$.

$$H^*_{dR}(BG/k) = \Lambda^*_k(\underbrace{\mathbb{D}(G)_{p\text{-tors}}}_{\deg 1}) \otimes_k \mathrm{Sym}^*_k(\underbrace{\mathbb{D}(G)/p}_{\deg 2}).$$

Rmk For $G = \mathbb{Z}/p$ or μ_p or α_p. $\mathbb{D}(G) \simeq k$ as $W(k)$-mod.

Thm (KLM) (1) The H-dR ss for BG/k always degenerates on E_2-page.

(2)* The H-dR ss degenerates on E_1-page

$\Leftrightarrow$ G lifts to a gp sch $/ W_2(k)$.

(3) $H^0(BG, \Omega^*_{BG/k}) = \Lambda^* H^0(BG, \Omega^1_{BG/k})$

all forms are closed (& non-transgressive.)

§2 Crystalline stories

Q How about $H^*_{crys}(BG/W(k))$?

as a lift of H-dR.

Thm (Mondal) $H^1_{crys}(BG/W(k)) = 0$.

$H^2_{crys}(BG/W(k)) \cong \mathbb{D}(G)$

Frob-equiv

Recall $R\Gamma_{crys}(-/W(k)) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \cong R\Gamma_{dR}(-/k)$.

$$\Rightarrow \quad 0 \to H^*_{crys}(-/W(k))/p \to H^*_{dR}(-/k) \to H^{*+1}_{crys,\, p\text{-tors}} \to 0$$

In order to understand / guess of the full

$$H^*_{crys}(BG) \quad \text{or} \quad R\Gamma_{crys}(BG).$$

Recall (Quillen, Illusie, …, Brantner–Mathew, Raksit):

$$L\mathrm{Sym}^*,\ L\Lambda^*,\ L\Gamma^* : \mathrm{Perf}(R) \longrightarrow D(R)$$

compatible w/ base change in R.

& Illusie's décalage:

$$L\Lambda^*(M[1]) \simeq L\Gamma^*(M)[*], \qquad * \in \mathbb{N}.$$
$$L\Lambda^*(M[-1]) \cong L\mathrm{Sym}^*(M)[-*].$$

Now if we let $M := \mathbb{D}(G)[-2]$

then $M \overset{L}{/} p \simeq \mathbb{D}_{p\text{-tor}}[-1] \oplus \mathbb{D}/p[-2]$.

Also, $L\Gamma^{\cdot}_k(M \overset{L}{/} p) := \bigoplus_{* \in \mathbb{N}} L\Gamma^*_k(M \overset{L}{/} p)$

$$\simeq \Big(\bigoplus_{* \in \mathbb{N}} \Lambda^*_k(\mathbb{D}_{p\text{-tors}})[-*]\Big) \otimes_k \Big(\bigoplus_{* \in \mathbb{N}} \mathrm{Sym}^*_k(\mathbb{D}/p)[-2*]\Big).$$

$\|$

$(L\Gamma^{\cdot}_{W(k)}(M)) \overset{L}{/} p$

$\rightsquigarrow$ Naïve guess (likely reasonable):

$$R\Gamma_{crys}(BG/W(k)) \simeq L\Gamma^{\cdot}_{W(k)}(\mathbb{D}(G)[-2]).$$

Answer: Yes and No!

Thm (KLM) (1)* As augmented $W(k)$-complexes, this is true.

(2) When $p>2$, $H^*_{crys}(BG/W(k)) \simeq H^*(L\Gamma^{\cdot}_{W(k)}(\mathbb{D}(G)[-2]))$.

Frob-equiv

Have

$$\begin{array}{ccccccccc} \text{augmented } R\text{-cplxes} & & A_\infty\text{-}R\text{-alg} & & & & & & \mathrm{DAlg}(D(R)) \leftarrow \mathrm{DdAlg}(D(R)) \\ & & \| & & & & & & \downarrow \\ & & & & & & & & \mathrm{CAlg}(D(R)) \\ \wr & & & & & & & & \| \\ \mathbb{E}_0\text{-}R\text{-alg} & \leftarrow & \mathbb{E}_1\text{-}R\text{-alg} & \leftarrow \cdots \leftarrow & \mathbb{E}_n\text{-}R\text{-alg} & \leftarrow \cdots \leftarrow & & & \mathbb{E}_\infty\text{-}R\text{-alg} \end{array}$$

(w/ worst commutativity).

(3) $R\Gamma_{crys}(B\mu_p/W(k)) \not\simeq R\Gamma_{crys}(B(\mathbb{Z}/p)/W(k))$
as $\mathbb{E}_1$-$W(k)$-alg.

§3 Even more general stories

Let (A, I) be a bdd prism,
then to any finite locally free comm gp sch $G/(A/I)$,
we get $R\Gamma_{\Delta}(BG/A)$.

<u>Thm</u> (KLM) $\exists$ increasing multiplicative exhaustive $\mathbb{N}$-indexed (motivic) filtration on $R\Gamma_{\Delta}(BG/A)$, functorial in G, satisfies base change in A, s.t.

$$Gr^{\bullet}_{mot} \cong L\Gamma^{\bullet}_{A}(\underbrace{\mathbb{D}_{\Delta}(G)[-2]}_{\text{by Anschütz-Le Bras.}})$$

as augmented graded derived A-alg + Frob.

<u>Q</u> Why stop at BG? (We have $B^2G, B^3G, \ldots$)

<u>Thm</u>* (KLM) $n \geq 1$. Suppose $G \hookrightarrow \mathcal{A}$ (formal) abelian sch $/(A/I)$,
$$R\Gamma_{\Delta}(B^nG, A) \simeq R\Gamma^{\bullet}_{A}(\mathbb{D}_{\Delta}(G)[-n-1])$$
as $\mathbb{E}_{n-1}$-$A[F]$-alg.

<u>Cor</u>* (KLM) G/k finite comm gp sch $/k$. Then
G lifts to gp sch $/W_2$ $\iff$ G lifts to comm gp sch $/W_2$.

<u>Thm</u> (KLM) $p = 2$.
$$H^*_{dR}(BG/k) = \underbrace{Sym^*_k(\mathbb{D}_{p\text{-tors}})}_{\deg 1} \otimes_k \underbrace{Sym^*_k(\mathbb{D}/p)}_{\deg 2}$$

(note: $\mathbb{D}p\text{-tor} \longrightarrow \mathbb{D}/p$ via $\alpha \longmapsto \alpha^2$

$$\begin{array}{ccc} \mathbb{D}p\text{-tor} & \longrightarrow & \mathbb{D}/p \\ \shortparallel & & \shortparallel \\ \mathbb{D}(G/p) & & \mathbb{D}(G_p\text{-tors}) \end{array}$$

with $\quad G/p \longleftarrow G \xleftarrow{\text{Frob}} G \longleftarrow G_p\text{-tors}$.

To explain relation to deform'n problem:

(1) D-I: $\mathcal{O}^{(1)} = \mathrm{Fil}_0^{\mathrm{conj}} \longrightarrow \mathrm{Fil}_1^{\mathrm{conj}} \longrightarrow \mathrm{gr}_1^{\mathrm{conj}} = \mathbb{L}_{X_0/W}[-1]$

$\rightsquigarrow \quad \mathbb{L}_{X_0/W} \longrightarrow \mathcal{O}^{(1)}[2]$

Corresp to lifting to W_2.

(2) Illusie: · lift BG to $W_2 \longleftrightarrow$ lift G as a gp sch to W_2.

· lift B^2G to $W_2 \longleftrightarrow$ lift G as a Comm gp sch to W_2.