Unipotent homotopy theory of schemes

Emanuel Reinecke

(Joint w/ Shubhodip Mondal).

§1 Background

$X/\mathbb{C}$ conn sch of fin type, $x \in X(\mathbb{C})$

$\rightsquigarrow \pi_i(X(\mathbb{C}), x)$.

Q $\exists$ def of $\pi_i(X(\mathbb{C}), x)$ that works over more general fields?

$k = \bar{k}$ field, X/k conn reduced sch of fin type $/k$, $x \in X(k)$

$\rightsquigarrow$ (Grothendieck) étale fund gp $\pi_1^{et}(X, x)$ (profinite gp.) $\rightsquigarrow$ fundamental gp sch $\pi_1^N(X, x)$ (Nori)

$\rightsquigarrow X/\mathbb{C}: \pi_1^{et}(X, x) = \pi_1(X(\mathbb{C}), x)^\wedge$.

$\rightsquigarrow$ (Artin-Mazur) étale homotopy gps $\pi_i^{et}(X, x)$. $\rightsquigarrow$?

Goal Define & study unipotent gp schs that acts as unipotent completion of ?.

Recall An affine gp sch G/k is unipotent if every fin dim'l repr $V \neq 0$ of G has $V^G \neq 0$.

E.g. $\mathbb{G}_a$, $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\} \subset GL_n$, $\underbrace{\alpha_p, \mathbb{Z}/p, \mathbb{Z}_p}_{\text{char } k = p > 0}$.

unipotent completion = map $G \longrightarrow G^{uni}$ universal w.r.t. maps to unipotent gp schs.

§2 Affine stacks (after Toen)

$X = \mathrm{Spec}\,\Gamma(X, \mathcal{O})$

DAlg_k^{ccn} = ∞-cat of cosimplicial k-alg / derived k-alg in deg ≥ 0.

$R\Gamma(X, \mathcal{O})$.

Have functor $\mathrm{Spec} : (\mathrm{DAlg}_k^{ccn})^{op} \longrightarrow \mathrm{Shv}(k)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ " fpqc sheaves of spaces /k.

$$R \longmapsto (A \longmapsto \mathrm{Map}(R, A)).$$

Def $X \in \mathrm{Shv}(k)$ is an affine stack if $X \simeq \mathrm{Spec}\,R^\cdot$ for some $R^\cdot \in \mathrm{DAlg}_k^{ccn}$.

Def $X \in \mathrm{Shv}(k)$, $x : * \longrightarrow X$. Then

- $\pi_i(X, x) :=$ fpqc sheafification of $(S \mapsto \pi_i(X(S), x))$.
- X is connected if $\pi_0(X) \simeq *$.

Toën's criterion for affineness

Thm (Töen) $X \in \mathrm{Shv}(k)$, $x : * \longrightarrow X$, conn.

Then X is an affine stack $\Longleftrightarrow$ $\pi_i(X, x)$ representable by unipotent gp schemes, $\forall i > 0$

§3 Unipotent homotopy grp schs

From now on: X = sch of fin type /k, $x \in X(k)$.

$\Rightarrow H^0(X, \mathcal{O}) \simeq k$.

Def $U(X) := \mathrm{Spec}\,R\Gamma(X, \mathcal{O})$ unipotent homotopy type of X.

$\pi_i^u(X) := \pi_i(U(X))$ homotopy gp schs of X.

First properties

(1) Hurewicz: TFAE: $\begin{cases} H^i(X,\mathcal{O}) \text{ trivial for } i \le n \\ \pi_i^u(X) \text{ trivial for } i \le n \end{cases}$

In this case, $\mathrm{Hom}(\pi_{n+1}^u(X), \mathbb{G}_m) \simeq H^{n+1}(X,\mathcal{O})$.

Ex If $H^i(X,\mathcal{O}) = 0$ $(i>0)$ (e.g. $X = \mathbb{P}_k^n$),

then $\pi_i^u(X) \simeq *$ $(i>0)$.

(2) Product formula:

$$\pi_i^u(X \times Y) \simeq \pi_i^u(X) \times \pi_i^u(Y).$$

(3) Birational invariance: $X \xrightarrow{f} Y$ birat morphism of sm proper varieties

$$\rightsquigarrow f_* : \pi_i^u(X) \xrightarrow{\sim} \pi_i^u(Y).$$

(essentially Chatzistamatiou - Rülling.)

(4) If X proper, char $k = p > 0$, then $\pi_i^u(X)$ is a profinite gp sch, $\forall i$.

§4 Comparison with previous homotopy theories

Thm If X proper $/k = \bar{k}$, char $k = p > 0$,

then the max'l proét quotient of $\pi_i^u(X)$ is naturally isom to

$$\pi_i^{et}(X)_p := \text{pro-}p\text{-finite completion of } \pi_i^{et}(X).$$

Map Using p-adic homotopy theory,

$$\pi_i(\mathrm{Spec}\, \underset{\substack{\cap \\ \mathrm{DAlg}_k^{ccn}}}{R\Gamma_{et}(X,k)}) \simeq \pi_i^{et}(X)_p$$

Note: $k \hookrightarrow \mathcal{O}$ gives $\mathrm{Spec}\, R\Gamma(X,\mathcal{O}) \longrightarrow \mathrm{Spec}\, R\Gamma(X,k)$

Nori unipotent fundamental gp schs

Nori: char $k = p > 0$. the unipotent completion of $\pi_1^u(X, x)$ is given by

$$\mathcal{U}(X, x) = \text{Tannaka dual of Tannakian cat of unipotent v.b. on } X.$$

Thm $\exists$ natural isom $\pi_1^u(X, x) \simeq \mathcal{U}(X, x)$.

§5 Formal group laws

(1) Abelian varieties

X = Ab var $/k$, $\dim X = g$.

Nori The Cartier dual $\pi_1^u(X)^\vee$ is the formal Lie gp obtained from the formal completion of $X^\vee$ at 0.

Prop The natural map $\mathcal{U}(X) \xrightarrow{\sim} \tau_{\leq 1}\mathcal{U}(X) = B\pi_1^u(X)$ an isom

$$\mathcal{U}(X) = \operatorname{Spec} R\Gamma(X, \mathcal{O})$$

In particular, $\pi_i^u(X, x) = 0$ for $i \geq 2$.

Pf idea Both affine stacks

$$\Rightarrow \text{STS } H^i(B\pi_1^u(X), \mathcal{O}) \longrightarrow H^i(\mathcal{U}(X), \mathcal{O}) = H^i(X, \mathcal{O})$$

(2) Curves.

X proper curve $/k$, $g = \dim H^1(X, \mathcal{O})$.

Nori $\pi_1^u(X)^\vee$ is a noncomm Lie gp of dim g.

Prop $\mathcal{U}(X) \xrightarrow{\sim} B\pi_1^u(X)$.

In particular, $\pi_i^u(X) = 0$ $(i \geq 2)$.

(3) Classification.

Q Is it possible to classify "comm alg structures" on $k \oplus k[-1] \in D(k)$?

$$\left\{\begin{array}{c}\text{augmented cosimplicial } k\text{-alg} \\ \text{structures on } k \oplus k[-1]\end{array}\right\} \qquad R \qquad\qquad R\Gamma(BE^\vee, \mathcal{O})$$

$$\Big\updownarrow 1:1 \qquad\qquad\qquad \Big\downarrow \qquad\qquad \Big\uparrow$$

$$\left\{\begin{array}{c}\text{(comm) formal gp laws} \\ \text{of dim 1}\end{array}\right\} \qquad \pi_1(\operatorname{Spec} R)^\vee \qquad\qquad E$$

weight $\hookrightarrow \mathbb{Z}_{>0} \cup \{\infty\}$ $\quad (k = \bar{k}$, char $k = p.)$

§6 Artin-Mazur formal groups

X proper $/ k = \bar{k}$, char $k = p > 0$, $\dim X = n \geq 1$.

Assume $H^i(X, \mathcal{O}) = 0$, $0 < i < n$, $g := \dim H^n(X, \mathcal{O})$.

Ex Calabi-Yau $\pi_i^u(x) = *$, $i < n$.

Similar as before, $\pi_n^u(x)^\vee$ is a comm formal Lie gp when $n > 1$

& a noncomm formal Lie gp when $n = 1$.

of dim g.

More concrete description:

Def / Thm (Artin-Mazur)

X as before. The contravariant functor

$$\Phi_X^n : \{\text{local art augm } k\text{-alg}\}^{op} \longrightarrow Ab$$

$$A \longmapsto \ker(H^n_{et}(X_A, \mathbb{G}_m) \to H^n_{et}(X, \mathbb{G}_m))$$

is prorep'ble by a formal Lie gp of dim g.

(called Artin-Mazur formal gp sch.)

Prop $\Phi_X^n \simeq (\pi_n^u(x))^\vee$ $\quad ((\pi_1^u(x)^{Ab})^\vee$ if $n = 1)$.

<u>Ex</u> X = weakly ordinary ($F^*: H^i(X,\mathcal{O}) \xrightarrow{\sim} H^i(X,\mathcal{O})$)

$\rightsquigarrow \Phi_X^n = \widehat{\mathbb{G}}_m$, i.e. $\pi_n^u(X) = \mathbb{Z}_p$.

$\rightsquigarrow \pi_i^u(X) \simeq \pi_i((S^n)_p^\wedge)$.

<u>Prop</u> X Calabi-Yau of dim ≥ 2 / $k = \bar{k}$, char $k = p > 0$.

Then $\mathcal{U}(X) \simeq \mathcal{U}(\Sigma^{n-1} B(\Phi_X^n)^\vee)$.

<u>Cor</u> X, Y as above.

$$D^b(X) \simeq D^b(Y) \underset{\uparrow}{\Rightarrow} \mathcal{U}(X) \simeq \mathcal{U}(Y)$$

+ work of Antieau-Bragg.

In particular, $R\Gamma(X,\mathcal{O}) \simeq R\Gamma(Y,\mathcal{O})$ as $\mathbb{E}_\infty$-alg $/k$.