Topological Reconstruction Theorems for Varieties (joint work with Max Lieblich, János Kollár, and Martin Olsson)

Will Sawin

Columbia University / CMI

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A controversial question

Q1

Do we *really* need rings to describe the structure of a variety?

Said better:

Q2

Can we reconstruct the scheme structure of a variety purely from its (Zariski) topological space?

A2: In certain circumstances, we can!

The classical topology

We will work with the Zariski topology, but first, what about the classical topology (over $\mathbb{C}$)?

Consider X a smooth, projective curve over $\mathbb{C}$, of genus g. Then $X(\mathbb{C})$ is a Riemann surface of genus g.

Underlying topological space determines X if $g = 0$, not otherwise.

In higher dimensions, similar: Topological space gives some useful information, but not nearly enough, except for very special varieties.

The Zariski topology: case of curves

For a curve of dimension 1 over a field k, the Zariski topological space consists of $\max(|k|, |\mathbb{N}|)$ closed points plus one generic point. A nonempty set is open if and only if it contains the generic point and all but finitely many closed points.

That is, we can reconstruct almost no information about the curve from its Zariski topology - not even the genus or whether it is affine or projective.

We only get information about the base field, and just a little (cardinality)!

Seems bad....

The Zariski topology: dimension theory

We do have some information in the Zariski topology.

Krull dimension of X: maximum length of an increasing chain of irreducible closed subsets of X.

We can distinguish some varieties of the same dimension, too: $\mathbb{P}^1 \times \mathbb{P}^1$ has two one-dimensional closed subsets that do not intersect, while $\mathbb{P}^2$ does not.

The Zariski topology: negative examples

(1) In characteristic p, totally inseparable maps can preserve the Zariski topology. The varieties defined by $z^p = f(x, y)$, for all polynomials f, have isomorphic Zariski topologies (and even étale topologies!).

Some of these are rational, others are general type - in general, they can have very different geometries.

(2) Let X and Y be two smooth projective surfaces of Picard rank one (e.g. $\mathbb{P}^2$) over finite fields (or the algebraic closure of a finite field). Then the Zariski topological spaces of X and Y are isomorphic.

Key lemma: Given irreducible curves $C_1, \ldots, C_n$ in X (or Y), and points $P_1, \ldots, P_m$, with at least one P_j on each curve C_i, there exists an irreducible curve D that intersects $C_1, \ldots, C_n$ only at $P_1, \ldots, P_m$.

Using this lemma, build an isomorphism step-by-step.

More data: The Picard group

For X an irreducible variety, a Weil divisor on X is a $\mathbb{Z}$-linear combination of irreducible codimension 1 closed subsets of X. (A purely topological notion.)

The Picard group of X is the group of Weil divisors, modulo the Weil divisors of rational functions on f.

We work with the Zariski topology of X, together with this equivalence relation on the group of Weil divisors. This will help to reconstruct the full geometry. We will be able to remove this additional data in some cases.

Our results

Theorem 1 (KLOS)

Let X be a proper, normal, geometrically integral variety of dimension at least 2 over an infinite field k.
Then X is completely determined as a scheme by its underlying topological space, together with its linear equivalence relation.

Theorem 2 (KLOS)

Let X be a proper, Cohen-Macauley, geometrically integral variety of dimension at least 3 over a finite k.
Then X is completely determined as a scheme by its underlying topological space, together with its linear equivalence relation.

Our results

Theorem 3 (KLOS)

Let X be a proper, normal, geometrically integral variety of dimension at least 2 over an uncountable algebraically closed field k of characteristic 0. Then X is completely determined as a scheme by its underlying topological space.

Theorem 4 (Kollár)

Let X be a projective, normal, geometrically integral variety of dimension at least 4 over a field k of characteristic zero, or dimension 3 over a field k finitely generated over $\mathbb{Q}$.
Then X is completely determined as a scheme by its underlying topological space.

Both Theorem 3 and Theorem 4 are proved by reduction to Theorem 1.

What can we understand about divisors?

A Weil divisor D is Cartier if and only if, for each $x \in X$, there is some Weil divisor $D' \sim D$ whose support does not contain x.

A Cartier divisor D is ample if and only if, for each $x, y \in X$, if x is not in the closure of y, then there exists $n \in \mathbb{N}$ and some effective Weil divisor $D' \sim nD$ whose support contains y but not x.

Strategy: Pick some ample divisor, and calculate its section ring! (+ tricks to reduce to the quasiprojective case)

The set of effective Weil divisors D' with $D' \sim nD$ is the projectivization of $H^0(X, \mathcal{O}_X(nD))$ (as a set).

Abstract projective spaces

The set of effective Weil divisors D' with $D' \sim nD$ is the projectivization of $H^0(X, \mathcal{O}_X(nD))$ (as a set).

We want to recover the *vector space* $H^0(X, \mathcal{O}_X(nD))$ from the *set* $\mathbb{P}(H^0(X, \mathcal{O}_X(nD)))$.

We need extra structure!

For Z in X a closed set, the set of $D' \sim nD$ with $Z \subseteq D'$ is the projectivization of a linear subspace of $H^0(X, \mathcal{O}_X(nD))$.

Question:

Does there exist a unique vector space V with bijection $\mathbb{P}(V) \cong \mathbb{P}(H^0(X, \mathcal{O}_X(nD)))$ that sends the set of divisors containing Z to the projectivization of a linear subspace?

Abstract projective geometry and the case of lines

Question:

Does there exist a unique vector space V with bijection $\mathbb{P}(V) \cong \mathbb{P}(H^0(X, \mathcal{O}_X(nD)))$ that sends the set of divisors containing Z to the projectivization of a linear subspace?

Remembering the classical theorem:

Theorem (Veblen-Young)

For a projective space $\mathbb{P}^n(k)$, there exists a unique vector space V with a bijection $\mathbb{P}(V) \cong \mathbb{P}^n(k)$ that sends lines of $\mathbb{P}^n(k)$ to two-dimensional subspaces of V.

We are led to consider the case when the set of divisors containing Z is a line. If we can find such a Z for every line L, we will have reconstructed $H^0(X, \mathcal{O}_X(nD))$.

When can we identify lines?

Let $L \subseteq \mathbb{P}(H^0(X, \mathcal{O}_X(nD)))$ be a line. When does there exist a closed subset $Z \subseteq X$ such that $Z \subseteq D'$ if and only if $Z \in L$?

L is the line between two points, corresponding to two divisors D_1, D_2. Under mild conditions, L is the pencil of divisors containing the closed subscheme $D_1 \cap D_2$. Good start!

But the closed subscheme $D_1 \cap D_2$ may not be reduced, in which case containing it as a subscheme and containing it as a subset are different.

For *generic* pencils, $D_1 \cap D_2$ is reduced.

Generic projective geometry

In fact we can check a little more, a generic line $L \in \mathbb{P}(H^0(X, \mathcal{O}_X(nD)))$ must be sent to a line $L' \in \mathbb{P}(V)$. We do this by building a maximal flag of linear subspaces and a corresponding descending chain of reduced closed subsets.

Question

For a projective space $\mathbb{P}^n(k)$ and a nonempty open set U of the Grassmanian of lines in $\mathbb{P}^n(k)$, does there exist a unique vector space V with a bijection $\mathbb{P}(V) \cong \mathbb{P}^n(k)$ that sends lines in U to two-dimensional subspaces of V?

Answer: Not quite, because we can modify the true bijection arbitrarily on a small closed subset.

But if we modify the statement to handle this, the answer is yes if k is infinite (but no if k is finite).

Idea: Go through the classical proof and think about genericity at each step!

From section spaces to section rings

We now have a vector space $H^0(X, \mathcal{O}_X(nD))$ and for most divisors D', we know there associated line in the vector space.

We want to construct the ring structure, i.e. multiplication $H^0(X, \mathcal{O}_X(nD)) \times H^0(X, \mathcal{O}_X(mD)) \to H^0(X, \mathcal{O}_X((n+m)D))$.

For the points where we know the associated divisor, this multiplication map is determined (up to scaling) by divisor addition.

Check that there are enough such points to uniquely determine the multiplication map everywhere (up to scaling), which determines the section ring, once we handle the scaling factors.

Associate points in X to ideals in this space to get a homeomorphism between X and Proj of the section ring. This completes the infinite field case.

But what about finite fields?

Over finite fields, a property that holds for generic lines could not hold for any lines defined over the base field.

We don't just need to understand the behavior of generic pencils - we need to count the pencils with bad behavior.

Probabilistic fundamental theorem of projective geometry

For each finite field k and $\delta > 0$ there exists $\epsilon > 0$ such that each bijection $f : \mathbb{P}^n(V) \to \mathbb{P}^n(W)$ of vector spaces over k that sends a proportion $1 - \epsilon$ of lines to lines, there exists a vector space isomorphism that agrees with f on a proportion $1 - \delta$ of points.

Proof: Borrow ideas from "linearity testing" in TCS to build an isomorphism of projective spaces by sampling random points.

Poonen (et. al.)'s Bertini

To complete the finite field argument, we want to show, for a proportion of pairs of $1-\epsilon$ of divisors D_1, D_2 in equivalence class nD, the intersection $D_1 \cap D_2$ is reduced.

Poonen showed: For X smooth, an explicit positive portion of divisors D in X are smooth.

Bucur-Kedlaya showed: For X smooth, a positive proportion of complete intersections $D_1 \cap D_2 \cap \ldots D_m$ are smooth.

But positive proportion may not be $1-\epsilon$. Problem: Any bad behavior at a given closed point can occur with positive probability.

Solution: Ignore closed points. For X smooth, a $1-\epsilon$ proportion of complete intersections $D_1 \cap D_2 \cap \ldots D_m$ are smooth away from a 0-dimensional set.

Poonen (et. al.)'s Bertini

Want to show: For a proportion $1-\epsilon$ of pairs of divisors D_1, D_2 in equivalence class nD, the intersection $D_1 \cap D_2$ is reduced.

For X smooth, we know that a proportion $1-\epsilon$ of complete intersections $D_1 \cap D_2$ are smooth away from a 0-dimensional subset. If $\dim X \geq 3$, $\dim D_1 \cap D_2 > 0$, this implies reduced.

For general X, stratify X by smooth schemes and apply Bucur-Kedlya to each. Conclude if $\dim X \geq 3$ that $D_1 \cap D_2$ is generically smooth for a proportion $1-\epsilon$ of pairs D_1, D_2.

Because X is Cohen-Macauley, $D_1 \cap D_2$ is Cohen-Macauley. Combined with generically smooth, this gives reduced.