Perverse filtrations, Hitchin fibrations, and compactified Jacobians

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(Joint with Davesh Maulik & Qizheng Yin).

<u>Overview</u>

$\boxed{\text{Perverse filtration}}$

$\nearrow$ abelian fib (w/ singular fibers): Hitchin systems, $P=W$ conj.

$\rightarrow$ Degenerating abelian var: DAHA, knots, ... (e.g. compactified Jacobians / affine Springer fibers).

$\searrow$ BPS invariants, etc.

§1 Perverse filtration

X sm proj var, L ample class,

$\rightsquigarrow L \circlearrowright H^*_{sing}(X, \mathbb{Q}) =: H^*(X)$.

- <u>Hard Lefschetz</u>: $L^i : H^{r-i}(X) \xrightarrow{\sim} H^{r+i}(X)$
- Relative case: $f : X \to Y$ proper

  Can assume X sm (not necessary yet).

<u>Relative HL</u> (a) (Perv filtration)

an increasing fil'n $P_0 \subset P_1 \subset \cdots \subset H^*(X)$.

(b) (Symmetry)

$L^i : {}^p\mathrm{Gr}_{r-i} H^*(X) \xrightarrow{\sim} {}^p\mathrm{Gr}_{r+i} H^*(X)$.

Idea def sing cohom via sheaf theory

$$H^*(X) = H^*(X, \mathbb{Q}) = \mathbb{H}^*(Y, Rf_*\mathbb{Q}).$$

$$P_\bullet := \mathbb{H}^*(Y, {}^p\tau_{\le \bullet}(-)).$$

Examples (1) $X = F \times Y \xrightarrow{f} Y$,

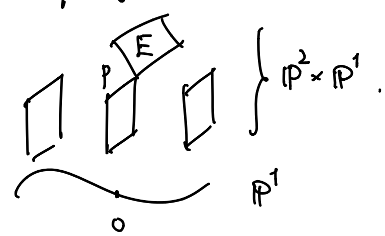
$H^*(X) = H^*(F) \otimes H^*(Y)$ (Künneth formula).

$P_k = H^{\le k}(F) \otimes H^*(Y)$.

(2) $f: X \to Y$ sm, $P_k = L_k$ Leray fil'n.

(3)

$$\begin{tikzcd} p \arrow[r, mapsto] & 0 \\ \mathbb{P}^2 \times \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \\ X = Bl_p(\mathbb{P}^2 \times \mathbb{P}^1) \arrow[u] \arrow[ur, bend right, "f"'] & \end{tikzcd}$$



note

	0	1	2	3	4
$Gr^L_\bullet$	*	0	*	0	*
$Gr^P_\bullet$	*	•	*	•	*

$[E] \in P_1 H^2$ $\quad$ $[E]^2 \in P_3 H^4 \setminus P_2 H^4$.

$\Rightarrow$ ⚠ Perv fil'n in general fails to be multiplicative w.r.t. "$\cup$". $\quad P_1 \cup P_1 \to P_3$.

§2 Higgs bundles

C, $g \ge 2$, $(n, d) = 1$.

$\mathcal{M}_{Dol}$ moduli of stable Higgs bundles $(\mathcal{E}, \theta)$

(hol symplectic mfd of dim $2n^2(g-1)+2$)

- $\mathcal{E}$ rk n, deg d.
- $\theta: \mathcal{E} \to \mathcal{E} \otimes \omega_C$ ($\mathcal{O}_C$-linear).

(a) **Hitchin fib** $h: \mathcal{M}_{Dol} \longrightarrow A = \operatorname{Spec} \mathcal{O}(\mathcal{M}_{Dol})$.

w/ A affine sp of $\dim = \frac{1}{2}\dim \mathcal{M}_{Dol}$.

$\rightsquigarrow (P_0 \subset P_1 \subset \cdots \subset H^*(\mathcal{M}_{Dol}))$.

(b) <u>non-abelian Hodge</u>

$$\mathcal{M}_{Dol} \xrightarrow[C^\infty]{\sim} \mathcal{M}_B = \{\pi_1(C) \to GL_n(\mathbb{C})\}/\sim$$

$(\rightsquigarrow H^*(\mathcal{M}_{Dol}) = H^*(\mathcal{M}_B).)$

"P=W" conj (de Cataldo - Hausel - Migliorini, 2010):

$$P_k H^*(\mathcal{M}_{Dol}) = \underbrace{W_{2k} H^*(\mathcal{M}_B)}_{\text{wt fil'n for } \mathcal{M}_B}.$$

- dC-H-M 2010: $n=2$ okay!

  <u>Q</u> $P_\bullet$ for Hitchin fib is multiplication?

- dC-Maulik-Shen 2019: $g=2$, $\forall n$, okay!

  In general, $\forall g$:

  "P = W" $\Longleftrightarrow$ $P_\bullet$ multiplicity.

- Maulik-Shen 2022:

  Hausel-Mellit-Minets-Schiffmann 2022 } $\forall g, n$ okay!

<u>Proof</u> $P_k = \underset{\substack{\uparrow \\ \text{Chern fil'n.}}}{C_k} = W_{2k}$

(c) Tautological classes

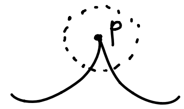
$$\begin{array}{ccl} & \mathcal{U} & p_{M*}[ch_k(\mathcal{U}) \cdot p_C^* \gamma] \in H^*(\mathcal{M}) \\ & & \qquad\quad \overset{\shortparallel}{c_k(\gamma)} \\ \gamma & \downarrow & \\ \searrow & C \times \mathcal{M} & C_k := \text{Span of } \prod c_{k_i}(\gamma_i), \ \sum k_i \le k. \end{array}$$

P=W is reduced to $P_k = C_k$.

$\uparrow$ Beauville decomp & Fourier tr.

§3 Curves and singularities

C curve w/ planar sing arith genus $= g$.



(e.g. $\mathbb{P}^1 = \tilde{C} \longrightarrow (C, p)$, unique.)

$\bar{J}_C = \{ \mathcal{F}$ rk 1 deg 0 tor-free sheaves on $C \}$

Compactified Jacobian, integral vcr of dim g.

$H^*(\bar{J}_C)$ rich structures.

<u>Upshot</u> $H^*(\bar{J}_C)$ carries a canonical perv fil'n (Maulik-Yun)

$\rightsquigarrow$ DAHA: Oblomkov-Yun

Link inv: Oblomkov-Rasmussen-Shende conj.

<u>Idea</u> $C \subset \mathcal{C}$ "large" enough

$$\begin{array}{ccc} C & \subset & \mathcal{C} \\ \downarrow & & \downarrow \\ 0 & \in & \Delta \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \bar{J}_C & \hookrightarrow & \bar{J}_{\mathcal{C}/\Delta} \\ & & \downarrow h \\ 0 & \hookrightarrow & \Delta \end{array}$$

$H^*(\bar{J}_C) = (Rh_* \mathbb{Q}_{\bar{J}_{\mathcal{C}/\Delta}})_0$

$P_\bullet := {}^p\tau_{\leq \bullet}$.

<u>Shende's proposal</u> Assume $C \ni p$ isolated w/ $\tilde{C} \simeq \mathbb{P}^1$

$\rightsquigarrow \beta$ link $C \cap S^3 \subset \mathbb{R}^4$.

$P_k H^*(\bar{J}_C) \overset{?}{=} W_{2k} H^*$(sth. only dep'd on link β).

$\Rightarrow P_\bullet H^*(\bar{J}_C)$ is multiplicative w/ "$\cup$" ? (*)

<u>Thm</u> (Oblomkov-Yun, 2017)

(*) is true for $x^p - y^q$, $(p, q) = 1$. (generators + DAHA).

<u>Thm</u> (Maulik-Shen-Yin, 2017)

(*) is true for any C.

<u>Idea of proof</u> (a) Arinkin's on $D^bCoh(\bar{J}_C)$ (Langlands)

- FM $\circlearrowright$ $D^bCoh(\bar{J}_C)$
- Convolution $F(\mathcal{E} \otimes \mathcal{F}) = F(\mathcal{E}) * F(\mathcal{F})$.

(b) Descent of motives / cohomology.

$$\mathfrak{F}, \mathfrak{F}^{-1} \curvearrowright H^*(\bar{J}_C).$$

(c) They are compatible w/ perv. filtrations

(Ngô support thm & Adam's operators).