Moduli stacks of crystals and isocrystals

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(Joint work in progress with Gyujin Oh.)

§ Introduction

X cplx manifold.

$$\begin{pmatrix} \mathbb{C}\text{-local system on } X, \\ \text{i.e. } \pi_1(X) \to GL_r(\mathbb{C}) \end{pmatrix} \xleftrightarrow[\text{Corr}]{\text{Riemann-Hilbert}} \begin{pmatrix} \text{vector bundles w/ integrable} \\ \text{connections on } X \\ (\mathcal{E}, \nabla: \mathcal{E} \xrightarrow{\nabla^2=0} \mathcal{E}\otimes\Omega^1_X) \end{pmatrix}$$

$$\mathcal{L} \longleftarrow (\mathcal{L}\otimes\mathcal{O}_X, 1\otimes d)$$

$$\mathcal{E}^{\nabla=0} \longmapsto (\mathcal{E}, \nabla)$$

"Betti side" "de Rham side".

Moduli Simpson: X sm proj alg var $/\mathbb{C}$

M_B (Coarse) moduli of rk r $\mathbb{C}$-LocSys on X

M_{dR} (Coarse) moduli of rk r v.b. w/ int conn on X

M_{Dol} (Coarse) moduli of rk r Higgs bundles.

$\rightsquigarrow$ $M_B^{an} \simeq M_{dR}^{an}$ analytic isom.

Nitsure: $\mathcal{U} \subset X \supset X - \mathcal{U}$ SNCD

constructed M_{dR} for log conn.

Today Analogous stacks $\mathcal{M}_{cris}$ & $\mathcal{M}_{isoc}$ for $X/\mathbb{F}_p$.

Setup p prime. $k = \mathbb{F}_q$ fin ext'n of $\mathbb{F}_p$. $W = W_{\mathbb{Z}_p}(k)$, $K = \mathrm{Frac}_{\mathbb{Q}_p} W$.

X sm alg var $/k$.

$\rightsquigarrow$ good cohom theory:

· $\ell \neq p$, ℓ-adic étale cohom

ℓ-adic LocSys on X, $X_{\bar{k}}$

· rigid cohom, (over)convergent (F-)isocrystals on X

w/ Frob str

Roughly, X sm proper. $X_W \to \operatorname{Spec} W$ sm proper lift.

OC isocrystals on X = v.b. with int conn on X_K^{an} ← generic fiber

satisfying certain rigid-analytic conv condition.

Q ∃ rigid-analytic stack of OC isocrystals on $\mathfrak{X}$?

Today Crystalline analogues $\mathcal{M}_{cris}$, $\mathcal{M}_{isoc}$

+ artin str Verschiebung endo.

§ Main thm & arithmetic motivation

X sm proper curve $/k$, $X_W \longrightarrow \operatorname{Spec} W$ lift.

$\forall S/W$, $X_S := X_W \times_W S \longrightarrow S$.

Thm (Oh-Shimizu)

(1) ∃ flat p-adic formal Artin stack $\mathcal{M}_{cris}$ over W

s.t. $\forall S$ p-adic flat formal sch / alg sp over W,

$\mathcal{M}_{cris}(S)$ = grpoid of rk r crystals of $\mathcal{O}_{X_S/S}$-mods

on big crys site $(X_S/S)_{CRIS}$.

~~together~~ with Verschiebung $V : \mathcal{M}_{cris} \longrightarrow \mathcal{M}_{cris}$

corresp. to "$\mathcal{E} \longmapsto F_r^* \mathcal{E}$".

(2) ∃ natural map $(\mathcal{M}_{cris})_\eta^{ad} \longrightarrow \mathcal{M}_{dR}(X_k; r)^{an}$

· its image $=: \mathcal{M}_{isoc}$ is an open substack

· $\mathcal{M}_{isoc}(k)$ is naturally identified w/ grpoid of rk r isocrystals on $(X/W)_{CRIS}$.

<u>Rmk</u> (1) Drinfeld, Bhatt–Lurie:

constructed $\mathcal{M}_{cris}$ from different perspectives.

(2) F-isocrystals on $(X/W)_{CRIS}$ $\underset{\substack{\uparrow \\ X \text{ proper}}}{=}$ OC F-isoc on X.

(3) $U \subseteq X$ dense open

$\rightsquigarrow$ Similar result holds for $((X, M_{X-U})/W)_{CRIS}$.

<u>Arith motivation</u> Langlands for GL_r:

X geom conn $/k$, F = fcn field of X, $U \subset X$.

Fix $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C} \simeq \bar{\mathbb{Q}}_p$.

$$\begin{tikzcd}
\left(\begin{array}{c}\text{Cusp autom rep's of } GL_r(\mathbb{A}_F) \\ \text{unram on } U \text{ w/ finite} \\ \text{order central char}\end{array}\right) \arrow[rr, leftrightarrow, "\text{Drinfeld } r=2", "\text{L. Lafforgue}"'] \arrow[dr, leftrightarrow, "\text{Abe}"'] & & \left(\begin{array}{c}\text{abs irred rk } r\ \ell\text{-adic} \\ \text{LocSys on } U \text{ w/ finite} \\ \text{order discriminants}\end{array}\right) \arrow[dl, leftrightarrow, "\ell\text{-}p"] \\
& \left(\begin{array}{c}\text{abs irred rk } r \text{ OC F-isoc on } U \\ \text{w/ fin order disc}\end{array}\right) &
\end{tikzcd}$$

<u>Drinfeld</u>: # of "such objects" $(r=2, X=U)$

$= q^{4g(X)-3} + \sum$ Weil numbers of wt $\in [0, 2(4g-3)]$

(about # of $\mathbb{F}_q$-pts of alg var.)

Rmk (1) Hongjie Yu generalized it to general r, $X = \mathcal{U}$.

(2) When $r=1$, $X = \mathcal{U}$,

$$\# \operatorname{Hom}(\underbrace{\operatorname{Im}(\pi_1(X_{\bar{k}}) \to \pi_1(X))}_{\operatorname{Pic}^\circ(X)(k)}, \bar{\mathbb{Q}}_\ell^\times) = \# \operatorname{Pic}^\circ(X)(k)$$

§ Construction idea of $\mathcal{M}_{cris}$

Start w/ $\mathcal{M}_{dR}$ Artin stack over W s.t. $\forall S/W$,

$\mathcal{M}_{dR}(S)$ = grpoid of rk r vbs with int conn for $X_S \longrightarrow S$.

Recall If S/k, $(\mathcal{E}, \nabla) \in \mathcal{M}_{dR}(S)$,

$\rightsquigarrow$ p-curvature $\psi(\nabla) = \psi(\mathcal{E}, \nabla) : \operatorname{Der}(X_S/S) \longrightarrow \operatorname{End}(\mathcal{E})$ p-linear

$$D \longmapsto (\nabla_D)^p - (\nabla_{D^p})$$

$\rightsquigarrow$ closed substack $\mathcal{M}_{dR,k}^{\psi\text{-nilp}} \subset \mathcal{M}_{dR,k} \subset \mathcal{M}_{dR}$

locus of ψ-nilp.

Observation S/W general

$(\mathcal{E}, \nabla)$ comes from crystal on $(X_S/S)_{CRIS}$

$\iff \psi(\mathcal{E}, \nabla|_{X_{S_k}})$ nilp

i.e.
$$\begin{tikzcd}
S \arrow[r] & \mathcal{M}_{dR} \\
\cup & \cup \\
S_k \arrow[r] \arrow[dr, dashed] & \mathcal{M}_{dR,k} \\
& \cup \\
& \mathcal{M}_{dR,k}^{\psi\text{-nilp}}
\end{tikzcd}$$

Constr'n / Prop $\mathcal{M}_{cris} :=$ p-dilatation of $\mathcal{M}_{dR,k}^{\psi\text{-nilp}}$ in $\mathcal{M}_{dR}$

i.e. flat p-adic formal Artin stack with

$$\begin{tikzcd}
\mathcal{M}_{cris} \arrow[rr] & & \mathcal{M}_{dR} \\
\cup & \circlearrowright & \mathcal{M}_{dR,k} \arrow[u, phantom, "\cup" description] \\
\mathcal{M}_{cris,k} \arrow[rr] & & \mathcal{M}_{dR,k}^{\psi\text{-nilp}} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

& universal among such.

Rmk Ogus, Abbes : for formal schs.

Obs $\Rightarrow$ $\mathcal{M}_{cris}$ satisfies desired moduli interpretation!

Verschiebung Rigidity of crystals :

$\mathcal{M}_{cris}(S)$ = rk r crystals on $(X_{S_k}/S)_{CRIS}$.

$$\begin{tikzcd}
X_{S_k} = X \times_k S_k \arrow[rr, "F_X \times id_{S_k}"] \arrow[d] & \circlearrowright & X \times_k S_k \arrow[d] \\
S \arrow[rr, "id"] & & S
\end{tikzcd}$$

$\rightsquigarrow$ map of topos $F_X : (X_{S_k}/S)^\sim_{CRIS} \longrightarrow (X_{S_k}/S)^\sim_{CRIS}$

$\rightsquigarrow$ F_X^* yields $V : \mathcal{M}_{cris} \longrightarrow \mathcal{M}_{cris}$.

E.g. rk 1 case : forget stacky issue (by rigidifying)

$$\begin{tikzcd}
0 \arrow[r] & H^0(X_W, \Omega^1_{X_W/W}) \arrow[r] & M_{dR} \arrow[r] & \mathrm{Pic}^0(X_W) \arrow[r] & 0 & /W \\
& & M_{dR,k} \arrow[u, phantom, "\cup" description] \arrow[r] & \mathrm{Pic}^0(X) \arrow[u, phantom, "\cup" description] & & /k \\
& & M_{dR,k}^{\psi=0} \arrow[u, phantom, "\cup" description] & & & \\
& \text{Cartier descent} \arrow[r] & \wr & \mathrm{Pic}^0(X^{(p)}) \arrow[uu, "F^*"'] \arrow[ull] & &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[dr, "F"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& \mathrm{Spec}\, k \arrow[r, "F_p"] & \mathrm{Spec}\, k
\end{tikzcd}$$

$\rightsquigarrow$ M_{isoc} = p-dilatation of $\mathrm{Pic}^\circ(X^{(p)})$ in M_{dR}

$\rightsquigarrow$ $V-1 : M_{isoc} \longrightarrow M_{isoc}$ grp homo.

Claim $\ker(V-1) =: M_{isoc}^{V=1}$ finite flat $/W$

$(M_{isoc}^{V=1})_k \xrightarrow{\sim} \mathrm{Pic}^\circ(X^{(p)})^{V=1}$

$\#(M_{isoc}^{V=1})(\mathbb{Q}_p) = \deg V-1$ on $\mathrm{Pic}(X^{(p)}) = \deg F-1$ on $\mathrm{Pic}^\circ(X^{(p)})$

$\updownarrow$

$F-1$ as dual isog.

$= \#\mathrm{Pic}^\circ(X^{(p)})(k) = \#\mathrm{Pic}^\circ(X)(k)$.

How about $\mathcal{M}_{isoc}$?

Need adic generic fiber

$(\mathcal{M}_{isoc})_\eta^{ad}$: (formal Artin stack$/W$) $\longrightarrow$ (adic Artin stack$/K$)

(i) for (R, R^+) affinoid $/(K, \mathcal{O}_K)$ with unique closed pt in $\mathrm{Spa}(R, R^+)$.

$$Z_\eta^{ad}(R, R^+) = \varinjlim_{\substack{R_0 \subset R^+ \\ \text{op bdd } W\text{-alg}}} Z(\mathrm{Spf}\, R_0)$$

(ii) Z Artin stack $/W$ $\rightsquigarrow$ $(Z_K)^{an}$ adic Artin stack

$\rightsquigarrow$ $(\hat{Z_p})_\eta^{ad}$

$\rightsquigarrow$ $(\hat{Z_p})_\eta^{ad} \longrightarrow (Z_K)^{an}$ (What does it look like?)

(iii) $Z = \mathbb{G}_m/W$, "$|t|_p = 1$" $\hat{\mathbb{G}}_m \hookrightarrow$ "$|t|_p \neq 0$" $\mathbb{G}_m^{an}$ $\Big\{$ general Z

(iv) $Z = B\mathbb{G}_m$, "$\hat{\mathbb{G}}_m$-torsors" $\longrightarrow$ "$\mathbb{G}_m^{an}$-torsors" $\Big\}$ mixture of those

$\begin{pmatrix}\text{moduli sp of}\\ \text{rk } 1\ \mathbb{Z}_p\text{-mods}\end{pmatrix}$ $\begin{pmatrix}\text{moduli sp of}\\ \text{dim } 1\ \mathbb{Q}_p\text{-v.s.}\end{pmatrix}$

$$\mathcal{M}_{isoc} := \mathrm{Im}\left(\mathcal{M}_{cris,\eta}^{ad} \xrightarrow[\text{immersion}]{\text{open}} ((\mathcal{M}_{dR})_p^\wedge)_\eta^{ad} \longrightarrow (\mathcal{M}_{dR,K})^{an}\right).$$