The Donagi-Pantev program for geometric Langlands on a curve of genus 2

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Introduction

The geometric Langlands correspondence was initiated for GL_2 in the work of Drinfeld, refined by Laumon, and then extended to GL_n by Frenkel, Gaitsgory, Vilonen, Arinkin, ...

It is related to the Langlands program obtained by Drinfeld, Lafforgue, Lafforgue, Ngô and others in arithmetic geometry, but the geometric version takes place in a complex setting.

Introduction

The correspondence says, in basic terms, that a local system with a given structure group over a Riemann surface C, leads to a perverse sheaf (or $\mathscr{D}$-module) on the moduli stack **Bun** of principal bundles for the Langlands dual group.

The main condition characterizing the perverse sheaf is that it's supposed to be an eigensheaf for the family of Hecke operators depending on the choice of a point in C.

By specializing to the open subset where the perverse sheaf is a local system, then applying the nonabelian Hodge correspondence, one gets a parabolic logarithmic Higgs sheaf.

Introduction

The Donagi-Pantev program envisions a beautiful description of the resulting objects, with the potential of providing a uniform generalization to all groups.

They have carried it out in a first case of rank 2 local systems on $\mathbb{P}^1 - \{5 \text{ points}\}$:

R. Donagi and T. Pantev, Parabolic Hecke eigensheaves
arXiv :1910.02357

The subject of this talk is my joint work with them, looking at a next case : rank 2 local systems over a compact curve C of genus $g = 2$.

Moduli stack of bundles

Let C be a smooth projective curve over the field of complex numbers. We'll later specialize to the case of genus 2.

If G is a reductive group, we have the moduli stack $\mathbf{Bun}_G$ of G-bundles over C. This has an open substack of semistable bundles, that admits a universal categorical quotient M_G, the moduli space of G-bundles on C.

Stable and very stable bundles

In turn, M_G has a Zariski open subset M_G^s of stable bundles, and this is a fine moduli space for the substack of stable bundles. The stack is in some cases a nontrivial gerb over the moduli space but we don't worry about that.

The **geometric Langlands program** constructs $\mathscr{D}$-modules on the stack $\mathbf{Bun}_G$. These are going to be smooth over some open subset of M_G^s, namely over the subset M_G^{vs} of *very stable bundles* introduced by Laumon :

- A bundle is very stable if it admits no nilpotent Higgs field.

The wobbly locus

The complement in M_G of the very stable open subset is a divisor that Donagi and Pantev named the *wobbly locus*.

The first main topological content of the geometric Langlands program is therefore the construction of local systems, i.e. representations of the fundamental group, over M_G^{vs}.

The Donagi-Pantev program looks to understand these local systems in terms of the nonabelian Hodge correspondence.

The wobbly locus

Let us change notation and denote by $X := M_G$ the coarse moduli space of semistable bundles. In our setting, G will be a group such as SL_n or PGL_n, specializing later to $n = 2$, although the construction is designed to work in general. Thus, X will be a moduli space of vector bundles on C with some condition on the determinant.

Let $W \subset X$ denote the wobbly locus. We will be looking for a construction of local systems on the quasiprojective variety $X - W$.

The Kobayashi-Hitchin correspondence

In order to study local systems over quasiprojective varieties, Takuro Mochizuki has developed nonabelian Hodge theory for this setting and beyond.

Theorem (T. Mochizuki's Kobayashi-Hitchin correspondence)

Suppose Y is a smooth projective variety and $D \subset Y$ a normal crossings divisor. Then the semisimple local systems on $Y - D$ correspond to the totally imaginary polystable parabolic logarithmic Higgs bundles on (Y, D) whose parabolic Chern classes c_1^{par} and c_2^{par} vanish.

The Kobayashi-Hitchin correspondence

Here, a parabolic Higgs bundle is a bundle E provided with filtrations and parabolic weights over the components of D, such that these filtrations have a common splitting at the singular points of D; together with a Higgs field $\varphi : E \to E \otimes \Omega^1_Y(\log D)$ respecting the filtrations.

The *totally imaginary* condition says that the residues of φ along divisor components should have totally imaginary eigenvalues. In our setup the eigenvalues will in fact be 0.

Conjugating by Kobayashi-Hitchin

Donagi-Pantev propose to conjugate the predicted geometric Langlands correspondence through the Kobayashi-Hitchin equivalence.

The input of the construction, that was originally a local system, is now viewed as a Higgs bundle (E, φ) on C.

Assuming it lies over a general point of the Hitchin base, the spectral curve $\widetilde{C}$ is smooth and (E, φ) corresponds by BNR to a line bundle L over $\widetilde{C}$.

Parabolic Higgs sheaves on (X, W)

On the $\mathbf{Bun}_G$ side of things, the $\mathscr{D}$-module $\mathscr{V}$ we are aiming to construct will be a local system over an open subset of the moduli space X.

As stated above, the divisor of singularities is $W \subset X$. Indeed, Laumon showed that the characteristic variety of $\mathscr{V}$ is the nilpotent cone itself.

This is expected to be the case for general structure groups too.

Parabolic Higgs sheaves on (X, W)

The completed objects we look for are thus Higgs bundles on X with a parabolic structure along W and Higgs field having logarithmic poles along the divisor.

A main difficulty is that W doesn't have normal crossings, so a method is needed to resolve singularities.

Luckily, it suffices to do so in codimension 2, by using Lefschetz theorems in Mochizuki's theory.

We'll stick to the notation (X, W), keeping in mind that resolutions are sometimes needed to make things well-defined.

Using the Kobayashi-Hitchin correspondence

In the higher dimensional situation there is still a *BNR correspondence* : logarithmic Higgs bundles correspond to sheaves on $T^*X(\log W)$ whose support, the *spectral variety*, is finite and dominant over X

It's logical to start by looking for the spectral variety. This will be a finite covering of X with a birational inclusion into the cotangent bundle T^*X.

Hitchin's moduli space

Donagi-Pantev propose using the **Hitchin fibration for C** in an ingenious and beautiful way to construct such a spectral covering.

So, our next main player is the Hitchin moduli space of Higgs bundles, provided with various structures such as the Hitchin fibration and the nilpotent cone.

Denote by $M_H(C, G)$ the moduli space of G-Higgs bundles over C. Those are pairs (P, φ) consisting of a principal G-bundle P and a section φ of the adjoint bundle with coefficients in the line bundle $K_C = \Omega^1_C$ of differential forms, known as the *Higgs field*.

Hitchin fibration and nilpotent cone

The invariant polynomials on the Lie algebra of G, when applied to φ, yield sections of symmetric powers of K_C. This defines the *Hitchin fibration*, a proper map from $M_H(C, G)$ to an affine space.

The *nilpotent cone* is the inverse image of the origin under the Hitchin fibration.

The associations $(P, \varphi) \mapsto (P, t\varphi)$ yield an action of $\mathbb{C}^*$ on the moduli space $M_H(C, G)$, and this collapses the whole moduli space to the nilpotent cone as $t \to 0$.

Spectral data

For $G = GL_n$ etc., a point in the base of the Hitchin fibration corresponds to a *spectral curve*

$$\begin{array}{ccc} \widetilde{C} & \hookrightarrow & T^*C \\ & {}_{\pi}\searrow & \downarrow \\ & & C \end{array}$$

and the fiber over that point is the set of Higgs bundles such that the spectrum of φ lies in $\widetilde{C}$ over each point of C.

When $\widetilde{C}$ is smooth, such a Higgs bundle is determined by $E = p_*(L)$ for the Beauville-Narasimhan-Ramanan data of a line bundle L on $\widetilde{C}$.

Components of the nilpotent cone

The nilpotent cone is a Lagrangian subset for the natural symplectic structure, and it is the union of several irreducible components.

Each component may be viewed as the downward flow applied to outgoing directions along a component of the fixed point set of the $\mathbb{C}^*$-action. This leads to an ordering, and the lowest piece is the moduli space of bundles $X = M_G$, where a bundle is viewed as a Higgs bundle with $\varphi = 0$.

Components and the wobbly locus

The wobbly locus may (most often) be described as the divisor in M_G consisting of points that are in the closure of the union of all the remaining components of the nilpotent cone. In other words, it is the set of points where the other components join onto the lowest one.

In the 5 point example the wobbly divisor had normal crossings in codimension two, but in general it can have other kinds of singularities. In the example we are looking at, these include cusps and tacnodes.

Components and the wobbly locus

The very stable points may also be characterized (thanks to the work of several people including Pauly, Pal, Peón-Nieto, Hausel, Hitchin) as the points where the upward flow is a closed subset, or equivalently where the Hitchin map is proper on the upward flow set.

Conversely, the wobbly points are those admitting an upward flow line that meets another component of the fixed point set.

Cotangent bundle

Recall that the moduli stack $\mathbf{Higgs}_G$ of G-Higgs bundles is the cotangent space of $\mathbf{Bun}_G$. Indeed, by Serre duality, the space of Higgs fields on a principal G-bundle P is

$$H^0(C, \mathrm{ad}(P) \otimes K_C) \cong H^1(C, \mathrm{ad}(P))^*,$$

in other words it is the dual of the tangent space of $\mathbf{Bun}_G$.

This is no longer quite true when we go to moduli spaces, however, $M_H(C, G)$ is birationally isomorphic in a natural way to $T^*X = T^*M_G$.

Hitchin fibers as spectral coverings

Therefore, one place to start looking for a spectral covering of X is to look for subvarieties in $M_H(C, G)$ that are finite coverings of X.

We have a nice source of these : the fibers of the Hitchin fibration.

A general fiber of the Hitchin fibration meets the inverse image of a general point of X in finitely many points. Thus, a Hitchin fiber fits what we're looking for.

As we'll see shortly, these relate quite naturally to the geometric Langlands program.

Geometric Langlands : Hecke eigensheaves

The specification of which objects correspond to each other under the geometric Langlands correpondence passes through the notion of "Hecke eigensheaves".

The *Hecke correspondence* is a diagram

$$
\begin{array}{ccc}
 & \mathbf{H} & \\
{}^{g}\swarrow & & \searrow^{h} \\
X & & X \times C
\end{array}
$$

parametrizing modifications of bundles by elementary transformations at points of C.

Geometric Langlands : Hecke eigensheaves

This leads to an action on constructible sheaves or $\mathscr{D}$-modules

$$\mathscr{V} \mapsto \mathrm{Hecke}(\mathscr{V}) := \mathbf{R}h_*(g^*\mathscr{V} \otimes \mathrm{Ker})$$

and an *eigensheaf* is a sheaf such that

$$\mathrm{Hecke}(\mathscr{V}) = \mathscr{V} \boxtimes E$$

for a local system E on C called the *eigenvalue*.

The predicted correpondence

The *Geometric Langlands conjecture* predicts that for a given local system E on C (viewed as a skyscraper sheaf on the moduli space of local systems), there exists a unique eigensheaf $\mathscr{V}$. This is a simplified statement and the formulation of a precise version is the subject of ongoing very complicated research.

But roughly speaking, we expect to have a construction that starts from a local system E over the curve C, and produces a $\mathscr{D}$-module $\mathscr{V}$ over $X \sim \mathbf{Bun}_G$.

In turn, $\mathscr{V}$ will basically consist of a local system over an open subset of X.

The predicted correpondence

In the good version of this setup, the “eigenvalue” E should be an ${}^{L}G$-local system for the Langlands dual group.

Various versions of the geometric Langlands conjecture have been proven for certain groups, specially of the form GL_n, SL_n, PGL_n, by Drinfeld, Laumon, Frenkel, Gaitsgory, Vilonen, Arinkin, ...

Less is known for more general groups. Furthermore, there has not been a satisfying geometric description of the basic operation of going from a local system on C to a local system over an open subset of X.

The program

The **Donagi-Pantev program** aims to describe the parabolic logarithmic Higgs sheaves on the moduli space X of G-bundles, corresponding to the local systems over open subsets of X that come from the geometric Langlands correspondence.

Statements going in the direction of their conjecture had also been thought of or suggested by others : Hausel, Faltings, Witten, ...

Creating BNR data : Electric-Magnetic Duality

Once we guess the choice of spectral covering being a Hitchin fiber, it remains to be understood how to define the spectral line bundle.

The theory of *Electric-Magnetic Duality* of Kapustin and Witten is a beautiful relationship between the data entering into geometric Langlands, and the BNR data that would be needed to define a Higgs bundle over X, at least birationally.

Let's look more closely at how these data are specified. Recall that an important part of the Langlands program is the relationship between Langlands dual groups.

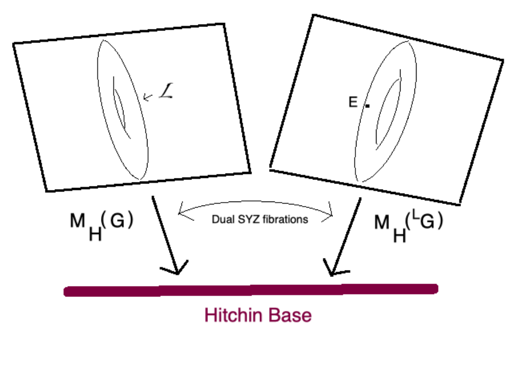
Electric-Magnetic Duality : Dual Hitchin fibrations

The fact that the Hitchin fibration reflects Langlands duality was first seen through the following picture due to Hausel, Thaddeus, Hitchin, Kapustin, Witten, Donagi, Pantev, . . . : the Hitchin fibers for Langlands dual pairs share the same base

$$\begin{tikzcd} M_H(G) \arrow[dr, "f"'] & & M_H({}^LG) \arrow[dl, "{}^Lf"] \\ & \mathbf{B} & \end{tikzcd}$$

where G is a reductive group, ${}^LG$ is its Langlands dual, and $\mathbf{B}$ is the Hitchin base common to both groups.

Electric-Magnetic Duality : Dual Hitchin fibrations



Theorem (Electric-Magnetic Duality)

The two Hitchin maps are generically dual SYZ-type torus fibrations.

The dual line bundle

The Langlands correspondence predicts that, given an ${}^LG$-local system E on C, there should be a $\mathscr{D}_X$-module $\mathscr{V}$ on X that is a Hecke eigensheaf for E.

Suppose that E corresponds to a point of $M_H({}^LG)$, in the fiber ${}^Lf^{-1}(b)$ over a general point $b \in \mathbf{B}$.

The fiber $Y_0 := f^{-1}(b) \subset M_H$ is the dual torus of ${}^Lf^{-1}(b)$, so the point E corresponds to a line bundle $\mathcal{L}$ over Y_0.

Donagi and Pantev propose to take $\mathcal{L}$ as input to construct a logarithmic parabolic Higgs sheaf on X that should correspond to the $\mathscr{D}_X$-module $\mathscr{V}$.

Hitchin fiber in the logarithmic cotangent bundle

To see how this works, let us recall that $M_H(G)$ is birationally the cotangent bundle T^*X. The inclusion $X \subset M_H(G)$ is just the zero-section.

This birational identification has poles along the wobbly W.

We obtain a morphism fitting into the following diagram, and include Y_0 in the picture :

$$\begin{tikzcd} Y_0 \subset M_H(G) \arrow[r] \arrow[dr] & T^*X(\log W) \arrow[d] \\ & X \end{tikzcd}$$

[some birational modification is needed as will be discussed below]

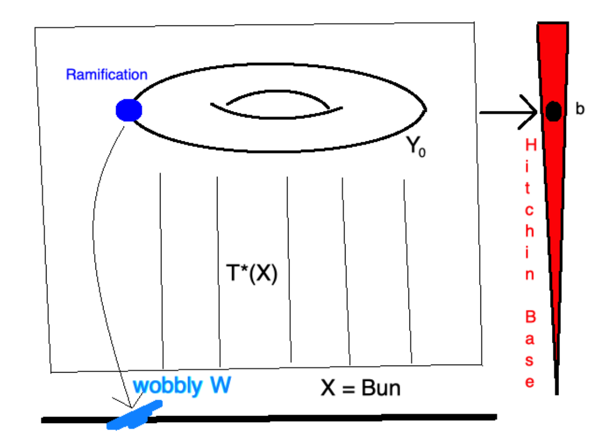
Construction of BNR data

In order to define a logarithmic Higgs sheaf on X with logarithmic pole along W, we should construct BNR-data consisting of a coherent sheaf on $T^*X(\log W)$ finite over X.

Typically BNR data comes from a subvariety—the *spectral variety*—provided with a line bundle.

In our case the line bundle will be a (modification of) the dual line bundle $\mathcal{L}$ seen above.

Construction of BNR data



Hitchin fibers are birationally finite coverings of X.

Construction of BNR data

Choice of a fiber $Y_0 := f^{-1}(b) \subset M_H$ of the Hitchin fibration provides a subvariety of $M_H(C, G)$, an ambient space that is birationally viewed as T^*X.

Globally, it turns out to give a subvariety of $T^*X(\log W)$, at least away from the codimension two singularities of W that might not be normal crossings and where the notion of logarithmic differentials is more subtle.

We'll use this as the spectral variety. The spectral line bundle is determined as above by electric-magnetic duality.

Logarithmic Higgs sheaf

We now have a subvariety $Y_0 \subset T^*X(\log W)$ together with a line bundle $\mathcal{L}$ on Y_0. These give therefore rise to a Higgs sheaf

$$(\mathcal{E} := \varpi_*(\mathcal{L}), \Phi) \text{ on } X$$

where $\varpi : Y_0 \to X$ is the projection.

The Higgs field has logarithmic poles along W :

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X(\log W).$$

The main idea

We may state the main idea of Donagi and Pantev rather simply :

Main Idea

The Hitchin fiber Y_0 serves to define the spectral variety for a Higgs sheaf on X, with spectral line bundle determined by electric-magnetic duality.

Some work needs to be done in order to make this spectral variety into something that will define a parabolic logarithmic Higgs sheaf $(\mathcal{E}, \Phi)$ on a normal-crossings resolution of (X, W), and then a main question is to show that it has vanishing Chern classes.

The Donagi-Pantev conjecture

Conjecture (Donagi-Pantev)

The Higgs sheaf $(\mathcal{E}, \Phi)$ defined above has a natural parabolic structure along the wobbly locus W, making it into the parabolic logarithmic Higgs sheaf corresponding to a local system on $X - W$ which is that of the $\mathscr{D}_X$-module $\mathscr{V}$ coming from the geometric Langlands correspondence.

Theorem (R. Donagi, T. Pantev)

This holds for $\mathbb{P}^1 - \{5 \text{ points}\}$.

Theorem (R. Donagi, T. Pantev, C.S.)

This holds for a compact genus 2 curve.

Moduli spaces on a curve of genus 2

The Langlands dual of $SL(2)$ is $PSL(2)$, but instead of considering projective bundles we'll look at bundles with fixed determinant of even or odd degrees.

Consider now a curve C of genus 2. Let $X(0)$ denote the moduli space of polystable rank 2 bundles with determinant $\mathcal{O}_C$. By Narasimhan-Ramanan $X(0) = \mathbb{P}^3$.

Let $\mathcal{X}(1) \xrightarrow{u} C$ be the family whose fiber $X_p(1)$ over a point $p \in C$ is the moduli space of stable rank 2 bundles with determinant $\mathcal{O}_C(p)$. Again by Narasimhan-Ramanan, $X_p(1)$ is a complete intersection of two quadrics in $\mathbb{P}^5$.

The Hitchin moduli spaces that we'll denote by M_H in these cases were the subject of an early paper by Previato and Van Geemen.

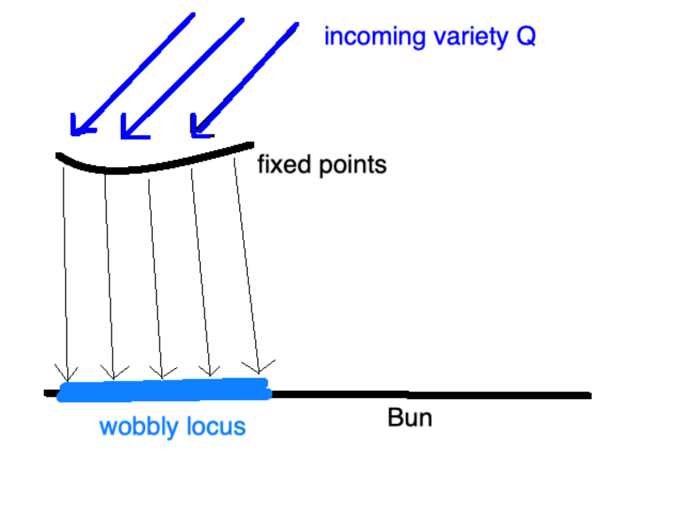
Not very stable = wobbly

Recall that the wobbly divisor $W \subset X$ is defined to be the set of polystable bundles that admit a nonzero nilpotent Higgs field, i.e. that are not "very stable".

Geometrically, the wobbly locus may usually[1] be characterized as the set of places over which the downward flow corresponding to the $\mathbb{C}^*$-action (also the same as the downward Morse flow for Hitchin's energy functional), has broken flow lines.

1. Our moduli space $X(0)$ is one of the rare exceptions to this rule.

The $\mathbb{C}^*$ flow in M_H



The wobbly locus in even degree

The wobbly locus inside $X(0)$ is given by

$$W(0) = K \cup T_1 \cup \cdots \cup T_{16} \subset \mathbb{P}^3 = X(0)$$

where K is the *Kummer surface* having 16 nodes, and T_i are the 16 *trope planes*. These planes and the nodal points of K form Kummer's 16_6-configuration : each plane passes through 6 points and each point is contained in 6 planes.

The trope planes intersect K in double curves, making a tacnode in a general transversal slice.

Odd degree—fixed point set

To describe the wobbly locus in odd degree we introduce the fixed-point set $\overline{C} \subset M_H$. This is the set of Higgs bundles of the form $L(p) \oplus L^{-1}$ with Higgs field $\varphi : L(p) \to L^{-1}(p+p')$ noting that $K_C = \mathcal{O}_C(p+p')$ where p' is the opposite to p under the hyperelliptic involution. The Higgs field has a unique zero, so we get

$$L(p) \cong L^{-1}(p+p'-q) \text{ or } L^{\otimes 2} = \mathcal{O}_C(p'-q).$$

It means that the set of these fixed points maps by a covering to the set of points $q \in C$. This covering is pulled back from the doubling map of the Jacobian, so it is $16:1$ and etale :

$$\overline{C} \xrightarrow{16:1} C.$$

The wobbly locus in odd degree

The wobbly locus in the odd degree moduli space may now be described as the image in $X_p(1)$ of the outgoing downward flow lines from points of $\overline{C}$. It turns out that the projectivized downward normal bundle is trivial of the form $\overline{C} \times \mathbb{P}^1$. Therefore :

- the wobbly locus $W_p(1) \subset X_p(1)$ is the image of a map

$$\overline{C} \times \mathbb{P}^1 \to X_p(1).$$

It turns out that the image has singular points : a cuspidal and a nodal locus at the first stage.

Non-well-definedness

In our discussion so far, we have ignored an important phenomenon. Namely, the projection $\varpi : M_H \to X$ is not well-defined, it is only a rational map. One should therefore make a birational modification to make it well-defined.

Let $Q \subset M_H$ be the incoming locus to the higher fixed-point set. We note that in both cases of degree 0 and degree 1, the fixed point set occurs at only one level : there are no curves in the nilpotent cone spanning between different higher fixed-point sets.

This is an important simplification specific to the genus 2 case.

Incoming locus

Write

$$Q = \{y \in M_H, \ \lim_{t\to 0} ty \in \mathrm{Fixed}^+\}$$

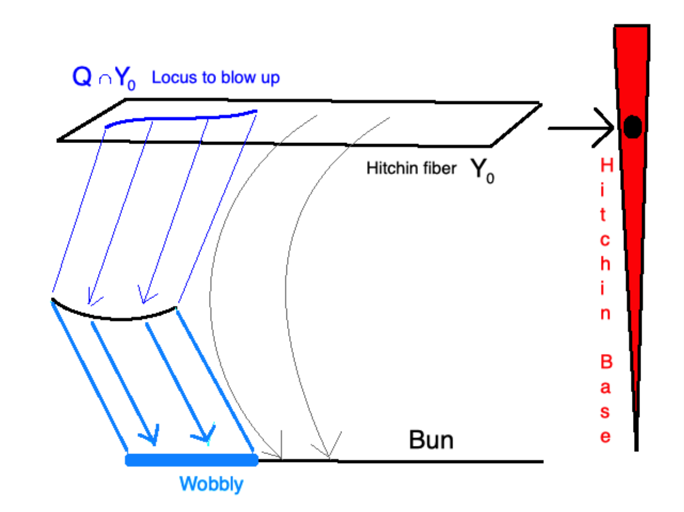
where $\mathrm{Fixed}^+ \subset M_H^{\mathbb{G}_m}$ is the union of fixed points that are not in X.

In degree 1 we have $\mathrm{Fixed}^+ = \overline{C}$ as was discussed above.

In degree 0 the Fixed^+ is a collection of the 16 uniformizing variations of Hodge structure of the form

$$\Phi : K_C^{1/2} \xrightarrow{\cong} K_C^{-1/2} \otimes \Omega^1_C$$

Intersection of the incoming variety with the Hitchin fiber



Resolution of the map

The projection map from M_H to X is not well-defined along Q. In our case, it becomes well-defined as a map

$$Y \subset \widetilde{M}_H \to X$$

where

$$\begin{array}{ccc} Y & \subset & \widetilde{M}_H \\ \downarrow & & \downarrow \\ Y_0 & \subset & M_H \end{array}$$

is the blowing-up of $Q \subset M_H$ and correspondingly $Q \cap Y_0 \subset Y_0$.

In higher rank and genus, Q will be singular so the resolution procedure will be more complicated. This is an important topic for further research.

Spectral curve

The point $b \in \mathbf{B}$ corresponds to a spectral curve

$$\begin{array}{ccc} \widetilde{C} & \hookrightarrow & T^*C \\ & {}_{\pi}\searrow & \downarrow \\ & & C \end{array}$$

In degree 1 the center $Q \cap Y_0 = \widehat{C}$ is a curve given as

$$\widehat{C} = \overline{C} \times_C \widetilde{C}.$$

In degree 0, $Q \cap Y_0$ consists of 16 points.

Exceptional divisor

Let $\mathrm{E} \subset Y$ denote the exceptional divisor. We have

$$\begin{array}{ccc} \mathrm{E} & \subset & Y \\ \downarrow & & \downarrow \varpi \\ W & \subset & X \end{array}$$

We consider the degree 0 line bundle $\mathcal{L}_0$ on Y_0, pull back to Y, and set $\mathcal{L} := \mathcal{L}_{0,Y}(\mathrm{E})$. Then

$$\mathcal{E} := \varpi_*(\mathcal{L})$$

is the bundle to look at on X.

Ramification

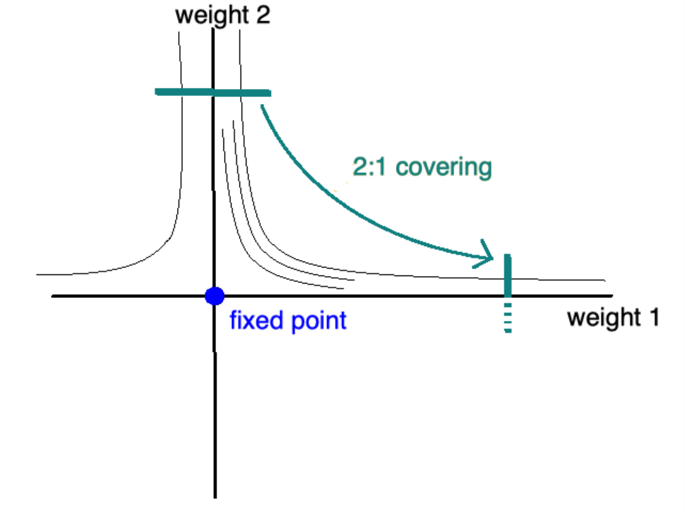
The projection from Y to X is ramified along the exceptional locus E (blow-up of $Q \cap Y_0$), hence it is ramified over $W \subset X$.

This part of the ramification remains fixed as we change the point in the Hitchin base ; there are other ramifications that move and don't contribute to the parabolic structure.

The reason for the ramification is that the weights of the $\mathbb{C}^*$ action at a fixed point are sometimes > 1. The incoming directions always have at least one piece with weight > 1. Flowing past a fixed point under these conditions induces a ramified map.

In turn, the ramification is what allows us to introduce a parabolic and/or logarithmic structure over W.

Ramification



Parabolic weights over $X_p(1)$

The bundle $\mathcal{E}$ has a natural quasiparabolic structure along $W_p(1)$, obtained from the fact that the exceptional divisor E is a locus of order two ramification of ϖ.

Chern class calculations show that one should place a parabolic weight of $\alpha = 1/2$ and this defines a parabolic bundle with logarithmic, strictly parabolic Higgs field along $W_p(1)$.

Parabolic weights over $X(0)$

Over $X(0)$, the exceptional divisor E is a union of 16 planes mapping to the trope planes T_i. No parabolic structure is put here ; the Higgs field is logarithmic and it will correspond to unipotent monodromy.

There is also going to be unipotent monodromy along the Kummer surface (as may be predicted by looking at the Hecke correspondence), so we don't need to put any parabolic weights along the general point of K.

Parabolic weights over $X(0)$

The location of the parabolic weights on $X(0)$ is therefore more subtle than for $X(1)$. The Kummer surface meets the trope planes in tacnodes. These need to be resolved in order to obtain a divisor with normal crossings.

Then, in order to get a truly logarithmic Higgs field together with the vanishing of the parabolic $\mathrm{ch}_2^{\mathrm{par}}$, there will be parabolic weights $1/2$ and $1/4,\ 3/4$ along the exceptional divisors obtained by blowing up (twice) the tacnodes.

Understanding the geometry of this specification is one of the technical difficulties.

Parabolic weights over $X(0)$—third construction

There is an alternate[2] collection of parabolic weights on $X(0)$ in which we put a parabolic structure with weight $\alpha = 1/2$ along the Kummer surface.

Using the theory of Goldman, Heu and Loray, this may be resolved by a natural double cover of $X(0)$ ramified along the Kummer, the moduli of parabolic rank 2 bundles on $\mathbb{P}^1$ with parabolic structure at the 6 points.

We also get vanishing parabolic Chern classes.

2. This construction was the one discussed in my talk in Kyoto several years ago.

Parabolic weights over $X(0)$—third construction

Although we don't currently have a full proof, it seems that the corresponding local systems on $X(0) - W(0)$ will be the ones associated by geometric Langlands to input $PGL(2)$-local systems of odd degree.

The Hecke transforms to $X(1)$ contain a singular locus that moves as a function of the point, indeed it is the Kummer K3 surface that appears under the terminology Σ in Chapter 5 of Griffiths and Harris.

For the present discussion let's get back to the Higgs bundles with trivial parabolic weights for $X(0)$.

Vanishing Chern classes

With the above definitions of parabolic structures, we calculate that the resulting parabolic sheaf $\mathcal{E}$ has vanishing parabolic Δ invariant, in other words, a tensorization by line bundle has vanishing c_1^{par} and c_2^{par}.

Some technical difficulties need to be overcome : the wobbly locus $W(1)$ has cuspidal singularities ; those may be removed by going to a covering with double ramification along $W(1)$ such that the covering is smooth, locally modelled on $\mathbb{C}^2/S_3 \to \mathbb{C}^2$. The parabolic weight $\alpha = 1/2$ disappears because of the order two ramification.

On $X(0)$ a closer analysis of the tacnodes is necessary.

Getting a local system

By Mochizuki's Kobayashi-Hitchin correspondence, the constructed parabolic Higgs sheaf therefore corresponds to a local system on $X - W$.

This is the Donagi-Pantev construction of a local system that is supposed to be the one coming from the geometric Langlands correspondence.

The next question is to prove that the resulting $\mathscr{V}$ satisfies the Hecke eigensheaf property.

The Hecke correspondence for our moduli spaces

Let's look here at the Hecke correspondence from $X(0)$ to $\mathcal{X}(1)$:

$$\begin{tikzcd} & \mathbf{H} \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X(0) & & \mathcal{X}(1) \end{tikzcd}$$

where

$$\mathbf{H} = \{(U, U', \alpha),\ \det(U) = \mathcal{O}_X, U \stackrel{\alpha}{\to} U' \to \mathbb{C}_p,\ p \in C\}.$$

The maps g and h are the first and second projections respectively.

Eigensheaf property

The geometric Langlands program says that the $\mathscr{D}$-module $\mathscr{V}$, in our setup having components $\mathscr{V}(0)$ on $X(0)$ and $\mathscr{V}(1)$ on $\mathcal{X}(1)$, should be characterized by the property of being an eigensheaf with respect to the Hecke operator, with eigenvalue given by the original local system E that we started with :

$$\mathbf{R}^{\cdot} h_*(g^*\mathcal{V}(0)) \cong \mathcal{V}(1) \boxtimes u^*(E)$$

where $u : \mathcal{X}(1) \to C$ was the projection.

Also vice-versa for the Hecke correspondence going in the other direction.

Dolbeaut higher direct image

The local system corresponding to $\mathscr{V}$ is created by constructing a parabolic logarithmic Higgs sheaf.

Since we are working in the Dolbeault world, we should therefore consider the higher direct image operation in the Dolbeault setting in order to verify the Hecke eigensheaf condition.

This was the motivation for our work with Donagi and Pantev on the Dolbeault calculation of higher direct images of Higgs bundles.

Dolbeault higher direct image

Consider in general a map $h : H \to X$ of relative dimension 1. Given a Higgs bundle (E, φ) on H, define the *Dolbeault complex*

$$DOL_{H/X}(E, \varphi) := \left(E \xrightarrow{\varphi_{H/X}} E \otimes \Omega^1_{H/X} \right)$$

over H.

The Dolbeault higher direct image vector bundle on X is

$$(E, \varphi) \mapsto F := \mathbf{R}^1 h_* DOL_{H/X}(E, \varphi),$$

and the global Higgs field upstairs leads to a Higgs field ϕ on this bundle.

Finite zeros

The Dolbeault construction can sometimes be calculated more explicitly.

If $\varphi_{H/X}$ has isolated zero eigenvalues in each fiber (generically the case), then the subscheme of zeros

$$\begin{array}{ccc} Z & \subset & \mathbb{P}(E) \\ & \searrow & \downarrow \\ & & X \end{array}$$

becomes the *spectral covering* for the Higgs bundle (F, ϕ) over X.

Finite zeros

More precisely, consider the quotient sheaf of the Dolbeault complex :

$$E \to E \otimes \Omega^1_{H/X} \to G \to 0$$

then $F = h_*(G)$.

This is because the finite zeros condition implies that the map in the Dolbeault complex is injective so the complex is quasiisomorphic to $G[-1]$.

Parabolic case

In practice, such as our present situation, (E, φ) has a parabolic structure and that needs to be taken into account in the Dolbeault complex. That theory is basically due to Saito, Sabbah and Mochizuki.

For our first paper with Donagi and Pantev, some work on understanding their theory was needed in order to produce a statement purely in the Dolbeault setting.

Fibers of h

Back to our situation, consider the map $h : \mathbf{H} \to \mathcal{X}(1)$. Following the above prescription, the first main question is to understand when the relative Higgs field of $g^*(E, \Phi)$ has a vanishing eigenvalue along a fiber.

Equivalently, we could consider the fiber $h^{-1}(x)$ over a point $x \in \mathcal{X}(1)$ and project it to $g(h^{-1}(x)) \subset X(0)$, then ask whether the restriction of the Higgs field Φ has a vanishing eigenvalue along here.

Interestingly enough, the images $g(h^{-1}(x)) \subset X(0) \cong \mathbb{P}^3$ are the lines in the famous *quadric line complex* discussed in Chapter 5 of Griffiths and Harris.

Higgs modifications

The zeros of the relative Higgs field have a satisfying description.

Lemma

The tautological form vanishes on the tangent direction to the Hecke curve, exactly when the Hecke modification is compatible with the Higgs field, or equivalently when it comes from a modification of the line bundle

$$0 \to L \to L' \to \mathbb{C}_{\tilde{p}} \to 0$$

at a point $\tilde{p} \in \widetilde{C}$ lying over $p \in C$.

The abelianized Hecke variety

This meets up with the *abelianized Hecke correspondence* defined by Donagi and Pantev :

$$\mathbf{H}^{\mathrm{ab}} := \{(L, L', \alpha),\ \ L \stackrel{\alpha}{\to} L'\}$$

where L and L' are line bundles over the spectral curve $\widetilde{C}$ such that $U = \pi_*(L)$ has determinant $\mathcal{O}_C$ and $U' = \pi_*(L')$ has determinant $\mathcal{O}_C(p)$ for some $p \in C$.

An alternate formulation is

$$\mathbf{H}^{\mathrm{ab}} := \{(L, \tilde{p}),\ \ \tilde{p} \in \widetilde{C}\}$$

where we put $L' := L(\tilde{p})$.

Identification with the zero-locus

The map $(L, L', \alpha) \mapsto (U, U', \alpha)$ with $U = \pi_*(L)$ and $U' = \pi_*(L')$, gives the vertical morphism in the diagram :

$$\begin{tikzcd}
 & & \mathbf{H}^{\mathrm{ab}} \arrow[dll] \arrow[d] \arrow[drr] & & \\
Y(0) \arrow[d] & & \mathbf{H} \arrow[dll] \arrow[drr] & & \mathcal{Y}(1) \arrow[d] \\
X(0) & & & & \mathcal{X}(1)
\end{tikzcd}$$

with compositions $g^{\mathrm{ab}} : \mathbf{H}^{\mathrm{ab}} \to X(0)$ and $h^{\mathrm{ab}} : \mathbf{H}^{\mathrm{ab}} \to \mathcal{X}(1)$.

According to the lemma, the subscheme of zeros of the relative Higgs field for $g^*(\mathcal{E}, \Phi)$ is precisely the abelianized Hecke $\mathbf{H}^{\mathrm{ab}}$.

The eigensheaf property on spectral varieties

The theorem on Dolbeault higher direct images says that there is going to be a natural line bundle $\mathcal{G}_H$ over $\mathbf{H}^{\mathrm{ab}}$ such that

$$\mathbf{R}^1 h_*(DOL_{\mathbf{H}/\mathcal{X}(1)}(g^*(\mathcal{E},\Phi)) = h^{\mathrm{ab}}_*(\mathcal{G}_H)$$

up to a choice of parabolic structure that needs to be calculated.

Basically $\mathcal{G}_H$ will be the pullback of our line bundle $\mathcal{L}$ on $Y(0)$ but again that needs to be modified over the parabolic divisors.

The eigensheaf property on spectral varieties

The Higgs field ϕ on the Dolbeault direct image will have $\mathbf{H}^{\mathrm{ab}}$ as spectral variety.
Geometric Langlands predicts that the direct image should decompose as an exterior tensor product of the constructed Higgs sheaf over $\mathcal{X}(1)$ with the Higgs sheaf corresponding to the original local system E.

(There is a finite-order twisting involved in $\mathcal{X}(1)$ related to the fact that we are looking at a fixed odd degree determinant rather than *PSL* objects.)

The eigensheaf property on spectral varieties

The eigensheaf property at the level of spectral varieties comes from the expression

$$\mathbf{H}^{\mathrm{ab}} \cong \mathcal{Y}(1) \times_C \widetilde{C}$$

whereby a point in $\mathbf{H}^{\mathrm{ab}}$ may be written $(L', \tilde{p})$ such that L' is a line bundle on $\widetilde{C}$ with $\pi_*(L')$ having determinant $\mathcal{O}_C(p)$ and $\tilde{p} \in \widetilde{C}$ with $\pi(\tilde{p}) = p$.

This was what Donagi and Pantev called the *classical limit of the Langlands correspondence* that they proved in great generality.

The full eigensheaf property

To go from the spectral version of the Hecke eigensheaf property to the full statement, we apply more precisely the theory of Dolbeault higher direct image to obtain a calculation of the higher direct image Higgs sheaf over the complement of a codimension 2 subset in the target moduli space. This is sufficient due to the Lefschetz-type theorems.

The full eigensheaf property

The decomposition as an exterior tensor product comes from looking at the locations of singularities and using the fact that irreducible local systems on product spaces dcompose as tensor products.

At this step, there is a question of showing that there is no singular contribution from certain apparent singularities along the Kummer K3 surface in $X(1)$.

[It is here that the third or alternate construction mentioned above is set apart : singularities along the Kummer K3 do occur. Presumably, in that case, when one makes a second Hecke transformation it should give an eigensheaf property with respect to the adjoint representation of $PGL(2)$; we haven't proven that.]

Comparison with Drinfeld's construction

Without appealing to a uniqueness result for Hecke eigensheaves, but rather using Dolbeault techniques, one may obtain the comparison with Drinfeld's original construction.

Theorem

The Hecke eigensheaves we construct here are the same as those constructed by Drinfeld-Laumon in this case.

This is proven by analyzing the spectral data underlying the nonabelian Hodge version of Drinfeld's Radon transform.